Application of Return Mapping Algorithm in Analysis of Nonlinear Kinematic Hardening Materials

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Abstract
This paper presents a return mapping algorithm for cyclic nonlinear kinematic hardening plastic constitutive model. The Armstrong - Frederick nonlinear kinematic hardening rule was used. A complete algorithm of the rate independent constitutive model for any desired strain or stress configuration was developed. The developed algorithm was applied for thick pressure vessel subjected to a cyclic internal pressure. The results were compared with results obtained by finite element method. This example demonstrated the accuracy of the method.

Keyword: Return mapping algorithm, nonlinear kinematic hardening, thick-pressure vessel.

Introduction
The life prediction of structures and structural components subjected to complex loading histories, which include repeated loading and unloading, requires an accurate representation of the material behaviour. The material models that approximate the behaviour of real materials involve complex computations which have become feasible only with the development of high speed computers and using numerical techniques. Therefore, the use of efficient procedure to determine the plastic strain is crucial. In order to determine the plastic strain at a current time step, the return mapping algorithm (RMA) is used in this investigation. RMAs [1, 2] represent a well established integration scheme to integrate the rate constitutive equations.

Jiang [3] studied cyclic elastic–plastic responses of tubes subjected to various loads and temperatures. He considered linear kinematic hardening rules. Nayebi and Abdi [4] studied the cyclic mechanical and thermal loading of a thick cylinder with creep effect. In this research the return mapping algorithm was applied for a thick cylinder under cyclic internal pressure. The algorithm was developed and the compatibility and the equilibrium equations were verified in each step. The obtained results were compared with the finite element results obtained by ANSYS code.

Return Mapping Algorithm

This method consists of an elastic trial and corrector steps as schematically represented by “Figure 1”. When the yield function is convex, that is, $f_n^{\text{trial}} > f_n$ at time step $n$, the elastic trial step is employed to characterize the plastic loading/unloading state of the material using the algorithmic Kuhn-Tucker conditions:

$$f_n \leq 0, \Delta \gamma_n \leq 0, \Delta \gamma_n f_n = 0 \quad (1)$$

At each time step, the yield function evaluated at the trial elastic step is used to determine whether or not the yield occurs. If the trial yield function is less than zero, then the material is assumed to be elastic or plastic but under elastic unloading. Otherwise, the material is subject to the plastic loading. The normality rule is used to define the plastic strain increment in the radial return algorithm. According to the normality rule, the plastic strain increment $\Delta \gamma_n$ is given by:

$$\Delta \gamma_n = \left( \frac{f_n^{\text{trial}} - f_n}{f_n^{\text{trial}}} \right) \Delta \gamma_n$$

Figure1: Illustration of the Return Mapping Algorithms.
strain is assumed to be in the direction normal to the yield surface defined in the deviatoric stress space and its magnitude is uniquely defined by the plastic multiplier. In order to determine the incremental value of the plastic multiplier $\Delta \gamma_n$ at the current time step $n$, the equation that ensures that the yield function is equal to zero is solved to determine $\Delta \gamma_n$. This equation is defined as linear in $\Delta \gamma_n$ in the case of the linear kinematic hardening. If a nonlinear kinematic hardening is assumed, then an iterative procedure such as Newton-Raphson method is used to determine the plastic strain at the current time step. Having determined $\Delta \gamma_n$, the plastic strains and the hardening parameter are updated, and the stress at the current time step is calculated using.

Mathematical Formulation

With the help of the deviatoric stress definition (eq. 2) and effective stress as:

$$\sigma^* = P \sigma, \tau = \sigma - X$$  \hspace{1cm} (2)

the Von-Mises yield function becomes

$$F = \frac{1}{2} \tau^T P \tau - \frac{1}{3} \sigma^2 \leq 0$$  \hspace{1cm} (3)

Using normality rule, the increment of plastic strain can be obtained as

$$d\varepsilon = \frac{dF}{d\sigma} = d\lambda P\tau$$  \hspace{1cm} (4)

So the plastic strain increment can be replaced in the accumulated plastic strain increment definition:

$$d\varepsilon_p = \frac{2}{3} d\lambda \sigma_y$$  \hspace{1cm} (5)

and the Armstrong-Freidrick nonlinear kinematic hardening reduced to:

$$dX = \frac{2}{3} C (Q\tau - \sigma Y) d\lambda$$  \hspace{1cm} (6)

Using the Euler implicit method, the incremental form of relations (3-6), stress-strain and back stress equations can be written as:

$$\tau_{n+1} = \sigma_{n+1} - X_{n+1}$$  \hspace{1cm} (7)

$$f_{n+1} = \frac{1}{2} \tau^T_{n+1} P \tau_{n+1} - \frac{1}{3} \sigma_y^2 \leq 0$$  \hspace{1cm} (8)

$$\varepsilon_{n+1}^p = \varepsilon_n^p + P \tau_{n+1} \Delta \lambda_{n+1}$$  \hspace{1cm} (9)

$$p_{n+1} = p_n + \frac{2}{3} \sigma_y^* \Delta \lambda_{n+1}$$  \hspace{1cm} (10)

$$\sigma_{n+1} = C (\varepsilon_{n+1}^p - \varepsilon_n^p) - CP \tau_{n+1} \Delta \lambda_{n+1}$$ \hspace{1cm} (11)

$$X_{n+1} = A^{-1} X_n + \Delta \lambda_{n+1} \frac{2}{3} C \sigma^* \Delta \lambda_{n+1}$$ \hspace{1cm} (12)

where $A = 1 + \frac{2}{3} C \sigma_y^* \Delta \lambda_{n+1}$ \hspace{1cm} (13)

Hopperstad and Remseth [5] proposed to divide the effective stress, $\tau$, to eigenvectors, $\nu_k$ and eigenvalues $\zeta_k$, as:

$$\tau_{n+1} = \sum_{k=1}^{3} \zeta_k \nu_k$$  \hspace{1cm} (14)

With replacing equations (11-13) and (14) in equation (7), the eigenvalues can be obtained:

$$\zeta_k = \frac{\alpha_k}{1 + \Delta \lambda \left( \frac{2}{3} C \sigma^* \Delta \lambda_{n+1} + \mu_k \right)}$$  \hspace{1cm} (15)

where $\alpha_k = \nu_k^T C^{-1} \nu_k^{\text{yield}}$ and $\nu_k^{\text{yield}} = C (\varepsilon_{n+1}^p - \varepsilon_n^p) - A^{-1} X_n$.

Putting equation (15) into equation (14) and then replacing effective stress in the equation (8), the yield function can be written as:

$$f_{n+1} = \frac{1}{2} \sum_{k=1}^{3} \mu_k \left[ 1 + \Delta \lambda \left( \frac{2}{3} C \sigma^* \Delta \lambda_{n+1} + \mu_k \right) \right]^{-1} \sigma_y^2 = 0$$  \hspace{1cm} (16)

that can be solved by the Newton-Raphson method.

Solution procedure for a thick cylinder analysis

The plane stress state in a thick cylinder under internal pressure is as equation (17):

$$\sigma = \begin{bmatrix} \sigma_r & \sigma_\theta & 0 \end{bmatrix}$$  \hspace{1cm} (17)

Using the equilibrium (18), compatibility (19) and stress-strain equations (20), the radial and tangential stresses as a function of the plastic strains can be obtained (21-22):

$$\frac{d\varepsilon_r}{dp} + \frac{\varepsilon_r - \varepsilon_\theta}{\rho} = 0$$  \hspace{1cm} (18)

$$\frac{d\varepsilon_\theta}{dp} + \frac{\varepsilon_\theta - \varepsilon_r}{\rho} = 0 \Rightarrow \varepsilon_r = \varepsilon_\theta + \rho \frac{d\varepsilon_\theta}{dp}$$  \hspace{1cm} (19)
To determine the response of a thick cylinder, the first step consists of dividing the cylinder wall into n layers and then applying loads, with the elastic assumption. Determining the maximum equivalent stress in every radius is carried out in a second step. Distribution of maximum equivalent stress determines the plastic regions. After determining the boundaries in the third step, the solution of the equation (16) lets to determine plastic multiplier in every layer. With the use of equations (7-13) the value of each parameter is updated. The flowchart is given in figure 3.

### Comparison of the Results

In order to compare the return mapping algorithm results with finite element results, a thick cylinder was modelled by ANSYS finite element code. 1176 elements of type PLANE183 were used. The proper mesh and the number of elements were selected according to the energy measurement convergence. The used mesh is given in Figure 2.

\[
\begin{align*}
\epsilon_r &= s_r - v (s_\theta + s_z) + e_\epsilon^R
\\
\epsilon_\theta &= s_\theta - v (s_r + s_z) + e_\epsilon^R
\\
\epsilon_z &= s_z - v (s_r + s_\theta) + e_\epsilon^R
\end{align*}
\]  
\tag{20}

\[
\sigma_r = \frac{-p}{r^2} + \left(1 - \frac{a^2}{r^2}\right) \frac{a^2}{b^2 - a^2} p
\]
\[
\frac{E}{2} \left[ (1 - \frac{a^2}{r^2}) \int_{\theta_1}^{\theta_2} \frac{\epsilon_r^p - \epsilon_\epsilon^R}{r} dr - \int_{\theta_1}^{\theta_2} \frac{\epsilon_\theta^p - e_\epsilon^R}{r} dr + \left( \frac{\epsilon_r^p - \epsilon_\epsilon^R}{r} \right)_{\theta_1, \theta_2} + 2 \frac{\epsilon_\epsilon^R}{r} \right]
\]  
\tag{21}

\[
\sigma_\theta = \frac{-p}{r^2} + \left(1 + \frac{a^2}{r^2}\right) \frac{a^2}{b^2 - a^2} p
\]
\[
\frac{E}{2} \left[ (1 + \frac{a^2}{r^2}) \int_{\theta_1}^{\theta_2} \frac{\epsilon_r^p - e_\epsilon^R}{r} dr - \int_{\theta_1}^{\theta_2} \frac{\epsilon_\theta^p - e_\epsilon^R}{r} dr + \left( \frac{\epsilon_\theta^p - \epsilon_\epsilon^R}{r} \right)_{\theta_1, \theta_2} + 2 \frac{\epsilon_\epsilon^R}{r} \right]
\]  
\tag{22}

Figure 2: The used mesh in finite element modelling.

Figure 3: Algorithm flow chart.
The method was applied to a thick cylinder under cyclic internal pressure with 100 mm inner radius and 120 mm outer radius. The geometry and material properties are given in Table 1. The constants of the material according to the Armstrong - Freidrick was given by [6].

Table 1. Geometrical and material properties of the thick cylinder under internal pressure

<table>
<thead>
<tr>
<th>( R_i ) (mm)</th>
<th>( R_o ) (mm)</th>
<th>( E ) (GPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>120</td>
<td>193</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \sigma_y ) (MPa)</th>
<th>( C ) (MPa)</th>
<th>( \gamma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>133</td>
<td>30000</td>
<td>60</td>
</tr>
</tbody>
</table>

The internal pressure was varied between 0 and the maximum pressure for different cycles and the using the RMA the distribution of stresses and plastic strains were obtained. The results were compared with the finite element method. “Figure 4” shows the comparison between the equivalent stress – strain values obtained by the RMA method and the FEM method results. The results showed that the return mapping method predicts with a good exactness the material behaviour.

The evolutions of the residual radial and tangential stresses as a function radius, for different cycles, are shown in “Figures 5, 6”.

The distribution of the radial and tangential strain for different cycles of loading are given in figures 7 and 8 as a function of radius.

Figure 4: Equivalent stress – strain in external radius.

Figure 5: Distribution of the residual radial stress in the cylinder thickness at different cycles.

Figure 6: Distribution of the residual tangential stress in the cylinder thickness at different cycles.

Figure 7: Distribution of the radial strain in the cylinder thickness at different cycles.
Figure 8: Distribution of the tangential strain in the cylinder thickness at different cycles

Conclusion

Return mapping algorithm was applied in cyclic loading analysis of a thick cylinder with nonlinear kinematic hardening material behaviour. The analytical relations were developed and this method can be extended to the more complex cyclic loading. In order to compare the results obtained by the proposed algorithm, the thick cylinder was modelled by finite element method. The results showed that the return algorithm has the necessary accuracy and is capable to model complex loadings.

References