A Fast Non-Iterative Algorithm to Predict Unsteady Partial Cavitation

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SUMMARY
A new algorithm to predict partial sheet cavity behavior on hydrofoils is proposed. The proposed algorithm models the unsteady partial cavitation using Boundary Element Method (BEM). In the proposed method the spatial iterative scheme is removed by means of a new approach determining the instantaneous cavity length. This iterative scheme is required in conventional algorithms to obtain the cavity length at each time step. Performance of the new algorithm for various unsteady cavitating flows with different reduced frequencies and hydrofoil geometries are investigated. Comparison between the obtained results using the proposed method and those of conventional ones indicates that the present algorithm works well with sufficient accuracy. Moreover, it is shown that the proposed method is computationally more efficient than the conventional one for unsteady sheet cavitation analysis on hydrofoils.

INTRODUCTION
Cavitation is usually observed in high-speed liquid flows around many obstacles such as impellers and hydrofoils. It is well understood that cavity flows will cause vibration, noise, damage and decline of efficiency in hydraulic systems. Partial sheet cavitation, cavitating tip vortex and supercavitating sheet are common types of cavitations that may occur when a high-speed liquid flows on a hydrofoil. Today, there are several computational modeling approaches to simulate cavitating flows around hydrofoils. Many of these models assume potential flow because of its simplicity and its suitable accuracy to analyze steady and unsteady cavitating flows around complex geometries [1-6]. Since BEM is known as a powerful computational method for potential problems, it is widely used by many researchers for cavitating flow analysis [1-7].

Analysis of cavitating hydrofoils using BEM have been published by, for instance, Lee and Kinnas [1], Pyo and Suh [2] and Salvatore and Esposito [7]. All these works are based on the 2-D closed partial cavity model of Kinnas and Fine [3], which consists of a so-called split panel approach and a pressure recovery region for the cavity closure. Vaz et al [4] reviewed and compared three different models for partial cavity flow modeling in the steady state. The above models are a fully non-linear model (FNL), a partially non-linear model with surface remeshing (PNL1), and a partially nonlinear model without surface re-meshing (PNL2). In FNL, the cavity surface is discretized using some boundary elements and the kinematic and dynamic boundary conditions are imposed on them [4]. But, in partially non-linear models one assumes a thin cavity and the boundary elements on the hydrofoil beneath the cavity surface are considered as the cavity boundary elements. In PNL1 some of the boundary elements are allocated to the cavity surface. These elements should be resized in accordance with the cavity length on each iteration step that results in resizing other boundary elements placed on the body. Therefore it is a surface remeshing procedure. On the other hand, in PNL2 the number of boundary elements assigned to the cavity is changed according to the cavity length and it is not required to resize the initial mesh when the cavity length is changed.

Vaz et al [4] and Vaz [5] concluded that the overall performance of PNL1 is much better than PNL2 and fully nonlinear models. Moreover, Vaz et al [4] concluded that PNL2 requires a larger number of panels than PNL1 to get numerical results with the same accuracy. However, robustness of PNL2 convinced Vaz [5] to use this model for three-dimensional steady cavitation analysis.

Regardless of which method (FNL, PNL1 or PNL2) is chosen, the cavity surface characteristics (cavity detachment point, cavity volume and length) are not known a priori and they should be obtained as part of the solution. Different criteria have been proposed in the literature to identify the location of cavity detachment point. The location of minimum pressure, the position where the surface pressure equals the fluid vapor pressure, the leading edge or the laminar separation point may be considered as the cavity detachment point. Moreover, a detachment condition known as the smooth separation condition by Brioullin and Villat (see Franc and Michel [8]) may be used to locate the detachment point. In addition, a closure condition is required in order to close the cavity surface. In steady state analysis a pressure recovery model has usually been used [5-6] as a closure condition while a dynamic boundary condition without any pressure recovery at the cavity end is usually applied for unsteady problems [5].

One of the main difficulties in the analysis of cavitating flows is determination of the free streamline (the cavity...
surface) on which the pressure is prescribed. Since the cavity surface is unknown an iterative procedure is required to determine it, see e.g. [3-6]. These iterative procedures clearly increase CPU time and computational costs especially for unsteady analysis.

In the present work a new approach for analysis of sheet cavitation is introduced which does not require any iteration. This new approach is based on the partially non-linear model without re-gridding previously developed for steady and unsteady partial sheet cavitation [5]. First, the mathematical model of partial cavity is presented for a hydrofoil in an unsteady flow. Next, the numerical model based on BEM is presented and the proposed algorithm is introduced. Then, the proposed method is applied for different geometries and various reduced frequencies. The obtained results are discussed and it is demonstrated that the proposed noniterative method can analyze various unsteady problems with sufficient accuracy. After that, the computational efficiency of the proposed method is investigated and shown that it is more efficient than the previous iterative methods. Finally, the paper is concluded with the abilities of the proposed noniterative method for unsteady partial cavitation analysis.

GOVERNING EQUATIONS

Consider a hydrofoil section with an attached cavity as shown in Fig. (1).

If the hydrofoil moves with $U_\infty$ along $x$ direction in a gusty flow described as $W_\infty(x,t) = W_0 + \Delta W f(x,t)$, the undisturbed flow velocity relative to a body fixed frame on the hydrofoil is

$$\vec{V}_\infty(x,t) = (U_\infty, W_0 + \Delta W f(x,t))$$  \hspace{1cm} (1)

where $W_0$, $\Delta W$ and $f(x,t)$ define the steady and unsteady parts of a possible gust in the flowfield. Using Eq. (1) for low angles of attacks one obtains

$$\frac{W_\infty(x,t)}{U_\infty} = \alpha_0 + \Delta \alpha f(x,t)$$  \hspace{1cm} (2)

where $\alpha_0$ is a steady state angle of attack and $\Delta \alpha$ is an amplitude about it.

Assuming potential flow, one can write

$$\nabla \Phi = V_\infty + \nabla \phi$$  \hspace{1cm} (3)

where $\Phi$ and $\phi$ are total and perturbation velocity potential, respectively. Outside the cavity, the perturbation potential satisfies Laplace’s equation, namely

$$\nabla^2 \phi = 0$$  \hspace{1cm} (4)

along with the boundary conditions that will be later discussed. Solving Eq. (4) subject to the corresponding boundary conditions, the perturbation potential is obtained and one can compute the pressure coefficient distribution using the unsteady Bernoulli equation that is [9]

$$\frac{2}{U_\infty^2} \frac{\partial \phi}{\partial t} + \frac{1}{2} \left(\nabla \Phi^2 - |V_\infty|^2\right) = -C_p$$  \hspace{1cm} (5)

where $C_p$ is the pressure coefficient defined as

$$C_p = \frac{p - p_\infty}{\frac{1}{2} \rho U_\infty^2}$$  \hspace{1cm} (6)

In the above relation, $p$ and $p_\infty$ are the perturbed and undisturbed flow pressures, respectively, and $\rho$ is the fluid density.

Equation (4) is a boundary value problem and needs boundary conditions on the entire flow boundaries. Flow boundaries can be considered as four surfaces illustrated in Fig. (1). They are the wet body surface $S_B$ (that part of the hydrofoil in contact with the liquid), the cavity surface $S_C$, the wake surface $S_W$ (the wake sheet behind the lifting hydrofoil) and the far boundary denoted as $S_d$.

The flow disturbances via the body motion should be diminished on $S_d$. In the other word

$$\lim_{r \to \infty} \nabla \phi = 0$$  \hspace{1cm} (7)

where $r$ is the distance from the origin of body’s frame of reference. On the wetted part of the body surface the fluid flow is tangent to the hydrofoil and a kinematic Neumann boundary condition is imposed, i.e.

$$\frac{\partial \phi}{\partial \vec{n}} = -V_\infty \hat{n}$$  \hspace{1cm} (8)

where $\hat{n}$ is the unit normal to the boundary pointing into the flowfield.

To impose the boundary condition on the wake surface, $S_W$, one knows that the vorticity generated at the trailing edge is shed into the wake and using Kelvin’s theorem it is convected along the wake surface with the free stream speed [6]. The generated vorticity at the trailing edge can be described using the familiar Kutta condition, namely

$$\left(\Delta \phi\right)_{TE} = \phi^T_{TE} - \phi^L_{TE}$$  \hspace{1cm} (9)
where \( \phi \) denotes trailing edge and \( \phi' \) and \( \phi'' \) correspond the wake upper and lower surfaces, respectively. Kelvin’s theorem results in,

\[
\Delta \phi_w (x, t) = \Delta \phi_{1x} \left( t - \frac{x - x_{1x}}{U_{\infty}} \right) \quad \text{on} \quad S_w
\]  

(10)

The questionable boundary condition is the one that should be imposed on the cavity surface. Since the cavity surface is not known a priori, two boundary conditions are considered; a dynamic boundary condition (DBC) and a kinematic boundary condition (KBC). The dynamic boundary condition on the cavity surface confirms that the pressure everywhere on the cavity surface is constant and equals to the vapor pressure. Using Eq. (5) it can be shown that this is equivalent to prescribing known values of \( \phi \) on the cavity, which satisfies [4-6],

\[
\phi (s, t) = \phi_0 (0, t) + \int_0^{\sigma} \left[ U_s \sigma + |V_n| - 2 \frac{\partial \phi}{\partial t} - V_n \right] ds
\]  

(11)

where \( \vec{s} \) is a curvilinear coordinate tangent to the cavity surface as shown in Fig. (1), \( \phi_0 \) is the perturbation potential at the cavity detachment point \( (s = 0) \), and \( \sigma \) is the cavitation number defined as

\[
\sigma = \frac{p_w - p_s}{\frac{1}{2} \rho U_{\infty}^2}
\]  

(12)

The unknown \( \frac{\partial \phi}{\partial t} \) in Eq. (11) is evaluated using the earlier values of \( \phi \) from the previous time steps [6].

The kinematic boundary condition guarantees the flow tangency on the cavity surface. It can be imposed using a partial differential equation for the cavity thickness, \( h \), as follows [4-6]

\[
U_e \frac{\partial h}{\partial n} = U_n - \frac{\partial h}{\partial t}
\]  

(13)

where \( U_e \) and \( U_n \) are velocity components tangential and normal to the cavity surface, respectively.

**NUMERICAL MODEL**

In the present study BEM is used as the numerical model to analyze the governing equation along with the imposed boundary conditions for partial cavitation analysis around hydrofoils. Applying Green’s theorem and imposing the boundary condition on \( S_w \) the equivalent boundary integral equation for Eq. (4) describing the perturbation potential at any point \( p \) on the hydrofoil and cavity surfaces gives Eq. (5) [9]. In Eq. (5) \( r \) is the distance from the point \( p \) to the boundary element \( ds \), \( c = 1/2 \) if \( p \) is on a smooth part of the surface and the integrals on \( S_B (t) \cup S_C (t) \) are in the sense of Cauchy principal value.

\[
2\pi c \phi_p (t) = \int_{S_w} \phi (t) \frac{\partial}{\partial n} (\ln r) ds
\]  

\[
= \int_{S_B (t) \cup S_C (t)} \ln r \frac{\partial \phi (t)}{\partial n} ds
\]  

\[
+ \int_{S_w} \Delta \phi_n (t) (\ln r) ds
\]  

(14)

In order to obtain an approximate solution for the boundary integral Eq. (14), the surfaces \( S_B \), \( S_C \) and \( S_w \) are discretized using small straight line elements. The value of \( \phi \) and \( \frac{\partial \phi}{\partial n} \) are assumed to be constant within each element. Therefore, the collocation method yields the following relation for each collocation point on the body [5]:

\[
\phi_j = \sum_{i=1}^{NB} A_{i} \phi_j + \sum_{i=1}^{NC} B_{i} \left( \frac{\partial \phi}{\partial n} \right)_j
\]  

\[
+ \sum_{i=1}^{NW} A_{i} \Delta \phi_j \quad , p = 1, 2, ..., NB + NC
\]  

(15)

where \( NB \), \( NC \) and \( NW \) are the number of elements on the wet body, cavity and the wake of the hydrofoil, respectively. Moreover,

\[
A_{pj} = \frac{1}{2\pi} \int_{S_j} \frac{\partial}{\partial n} (\ln r) ds_j
\]  

(16)

\[
B_{pj} = -\frac{1}{2\pi} \int_{S_j} (\ln r) ds_j
\]  

(17)

are influence coefficients and \( s_j \) is the surface of \( j^{th} \) element.

For the first term on the right hand side of Eq. (15) \( \phi_j \) is known on the cavity surface from boundary condition described by Eq. (11). On the other hand \( \frac{\partial \phi}{\partial n} \) on the wet body is determined using the tangency boundary condition represented by Eq. (8). The third term on the right hand side of Eq. (15) is also known because \( \Delta \phi \) can be obtained using the Kutta condition and Kelvin’s theorem denoted by Eq. (10). Therefore, there are \( NB \), \( NC \) and \( NW \) unknowns for \( \phi \) on the wet body and \( NC \) unknowns for \( \frac{\partial \phi}{\partial n} \) on the cavity surface that are obtained using Eq. (15).

Because the cavity surface is not known a priori, imposing the boundary condition on the cavity surface is not a clear and straightforward procedure. One doesn’t know where the start and the end points of cavity are. Therefore, the partial cavity
problems are usually analyzed using iterative algorithms. For example let \( h(s) \) denotes the cavity thickness. It should be equal to zero at \( s = l \), where \( l \) is the cavity length. In a conventional unsteady algorithm a spatial iterative scheme is used at each time step to find the correct value of \( l \). The spatial iterative scheme is begun with an initial guess for the cavity length. Next, Eq. (15) with corresponding boundary conditions is solved and \( \phi \) and \( \frac{\partial \phi}{\partial n} \) distribution on the wet body and cavity surface are determined, respectively. Then the cavity thickness, \( h(s) \), is calculated using Eq. (13). If the cavity thickness at \( s = l \) converges to zero \([5,6]\), namely:

\[
| h(l) | \leq \varepsilon
\]

where \( \varepsilon \) is an accepted small error, the solution is converged and the guessed length is the correct one. Of course \( | h(l) | = \varepsilon > \varepsilon \) at each iteration step if the cavity length is smaller or larger than the converged value.

In the present work, a noniterative algorithm is proposed which eliminates the above time consuming spatial iterative procedure. The idea of developing the current algorithm is based on the behavior of cavity thickness, \( h(l) \), versus the guessed cavity length in a steady flow. Figure 2 illustrates the cavity thickness at \( s = l \) for different guesses of cavity lengths \( l \) for various angles of attacks and cavitation numbers. The cavity detachment point is considered at the leading edge. Obviously the correct cavity length for each case is the one for which \( h(l) \) is zero. As is shown in the figure the error near the solution \( (\varepsilon \approx 0) \) is quite linear for all of these cases. Moreover, for each cavitation number the error slope near the solution is nearly the same for all angles of attacks. Table 1 represents the error slopes for those angles of attacks and cavitation numbers of Fig. 2. One can assume a slope of -0.08 and -0.10 for \( \sigma = 0.9 \) and \( \sigma = 1.0 \), respectively for NACA16-006 as depicted in table 1. For other geometries similar data can be obtained. It should be noted that the above data are for steady cases and therefore they are computed once and are used for the proposed unsteady algorithm.

Now, let’s see how the proposed algorithm is applied for unsteady partial cavitation analysis. In the current algorithm a cavity length for a steady flow with an average loading condition (average angles of attacks of the corresponding unsteady problem) is calculated first. Next, Eq. (15) with boundary conditions (7), (8), (10) and (11), is solved that problem. The procedure modifies the flow velocity adjacent the cavity end so that the zero velocity condition is fulfilled there. Accordingly, it is expected that the current approach is relatively improved. Let \( g(l') \) denotes a pressure coefficient time variations. It was obtained that the results of proposed algorithm have nearly the same amplitude but they have a small shift in comparison with those of the conventional iterative approach. More assessment showed that the main reason for aforementioned shift is the pressure distribution behavior adjacent the reattachment region of cavity. The pressure coefficient distribution at the instant of fourth cycle of hydrofoil oscillation is shown in Fig. 3. As is depicted in Fig. 3 one can observe that there is a large difference between the pressure coefficients computed by the two methods near the cavity end. Because the end point of cavity should be stagnation point \([3-6]\), the pressure coefficient should be near one there. Thus, the results obtained by the conventional iterative method are more consistent with the physics of the flow.

![Figure 2: Cavity heights (\( \varepsilon \)) vs. different guesses of cavity lengths for NACA 16-006 (c is the chord of hydrofoil)](image)

<table>
<thead>
<tr>
<th>Angle of attack (AOA)</th>
<th>3.5</th>
<th>4</th>
<th>4.5</th>
<th>Average</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cavitation number (( \sigma ))</td>
<td>0.9</td>
<td>-0.0827</td>
<td>-0.0761</td>
<td>-0.0685</td>
</tr>
</tbody>
</table>

![Figure 3: Pressure coefficient: conventional method and proposed method without pressure recovery role](image)
recovery function by which the flow velocity is corrected as
\[
V^* = [1 - g(s')] V
\]  
(19)
Where \( V \) is the total velocity, \( V^* \) is the corrected velocity and \( g(s') \) is 
\[
g(s') = \begin{cases} 
\frac{s_j}{s} - s_j & l_j \leq s' \leq l_j + \Delta l \\
0 & \text{Otherwise}
\end{cases}
\]  
(20)
in which \( s_j \) is the distance between the reattachment point and the collocation point on its neighbor element. Moreover, \( s' \) is the distance between the reattachment point and the end point of next elements and \( \Delta l \) is a suitable region near the reattachment point. Based on the authors’ numerical experiences \( \Delta l = 0.3 \ell \) is an appropriate choice for various problems. If the algorithm is modified based on the above procedure, the corresponding pressure coefficient distribution of Fig. 3 will be obtained as shown in Fig. 4. As is illustrated in Fig. 4 the proposed algorithm results approach the ones of conventional iterative method with a small acceptable error.

Figure 4: Pressure coefficient: conventional method vs. proposed method with (w/) and without (w/o) pressure recovery after reattachment point

RESULTS AND DISCUSSIONS
NACA16-006 and NACA16-009 hydrofoils are considered as test cases to verify the proposed algorithm in unsteady partial cavitation flows. The results are presented for \( \sigma = 1.0 \) which is the most conventional one for partial cavitation problems [3-6]. The unsteady flow described by Eq. (2) around NACA16-006 is considered as
\[
W_C x, t \Omega = 4 + 0.5 \sin(\Omega T)
\]  
(21)
where \( T = \frac{2U_{\infty} \ell}{c} \) is the nondimensional time, \( \Omega = \frac{\omega c}{2U_{\infty}} \) is the reduced frequency and \( \omega \) is frequency of unsteady flow motion. For a time interval equivalent to 16 cycles of oscillations, spatial and temporal discretizations are considered as those of Vaz [5], i.e., 400 panels on hydrofoil with cosine distribution and 10 time steps per cycle. The first step is to calculate cavity length in the corresponding steady flow for average loading (\( \alpha_c = 4 \% \)). It is about 0.4 \( \ell \) as shown in Fig. 2.

First, the ability of the proposed method for analysis of unsteady cavitating flows with various reduced frequencies is investigated. Figures 5 illustrate time changes of cavity length and lift coefficient for NACA16-006 hydrofoil. In the corresponding unsteady partial cavitation flows reduced frequencies of 0.5, 1 and \( \frac{\pi}{2} \) are considered. It is observed that the results of proposed method are in accord with the conventional ones. The maximum relative error is about 6% for both cavity length and lift coefficient predictions. However, there are relatively large differences for a few initial time steps where nonphysical peaks exist due to the time derivative term in relation (11) [5]. Fortunately, these transients decay quickly.

Next, the capability of the proposed method is discussed for other geometries. A NACA16-009 hydrofoil is considered as another geometry and the obtained results of the present method are compared with those of the conventional one. In the present study it is assumed that \( \sigma = 1 \), and \( K = \frac{\pi}{2} \).

Moreover, steady state analysis results in the error slope of -0.085 for this hydrofoil. Figures 6 shows time changes of cavity length and lift coefficient on this section. As are illustrated in the figures good agreement exists between the results of proposed and conventional ones. Maximum relative errors are 6% and 7% for cavity length and lift coefficient calculations, respectively. Based on the obtained results one concludes that the proposed method can be applied for other hydrofoil geometries with sufficient accuracy. In other words the proposed method is a sufficiently accurate and general approach for unsteady partial cavitation analysis on hydrofoils.

Figure 5: Time changes of cavity length (left) and lift coefficient (right) for NACA 16-006 with various reduced frequencies
Finally, the computational efficiency of the proposed method is discussed through comparison between the CPU times of the proposed method and the conventional iterative one. Let’s define time efficiency of the present method as

$$\eta = \frac{T_p - T_c}{T_c} \times 100 \quad (22)$$

where $T_c$ and $T_p$ are CPU times of the conventional and present methods, respectively. Table 2 depicts CPU times in seconds for analysis of unsteady cavitating flows with various reduced frequencies on NACA 16-006 section via the conventional and proposed methods. The presented results are based on numerical computations using a Dual-Core-2100 MHz with 2-GB RAM computer. As is depicted in the table 2, the proposed method is much more efficient than the conventional iterative one. It is observed that the present method has a time efficiency of more than 80% which is an excellent efficiency for a computational approach. Thus, one can consider the present method as a fast non-iterative algorithm.

Table 2. Comparison of CPU time (s) in different situations

<table>
<thead>
<tr>
<th>$K$</th>
<th>$T_c$</th>
<th>$T_p$</th>
<th>$\eta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>49.3</td>
<td>8</td>
<td>84%</td>
</tr>
<tr>
<td>1</td>
<td>42</td>
<td>7.83</td>
<td>82%</td>
</tr>
<tr>
<td>$\pi/2$</td>
<td>62.4</td>
<td>7.55</td>
<td>88%</td>
</tr>
</tbody>
</table>

CONCLUSION

The proposed numerical algorithm can accurately and efficiently predict partial cavity effects on hydrofoils in unsteady flows. Using the proposed method the cavity length and lift coefficient can be accurately predicted without any iteration requirement at each time step. An algebraic and simple relation is proposed to modify the pressure distribution close to the cavity reattachment point. However, the proposed method prettily works for several reduced frequencies and different geometries. Comparison between the results of conventional and proposed methods shows that the maximum relative difference is less than 7%. Moreover, the obtained results show that the proposed method has excellent time efficiency greater than 80%. Having the above characteristics, one concludes that the proposed method is a powerful numerical approach for analysis of unsteady partial cavitation problems.

REFERENCES