The Optimization of a Full State Feedback Control System for a Model Helicopter for Longitudinal Movement

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Abstract

In this paper, a closed loop pole location and the speed of response for a model helicopter in longitudinal movement is considered. An optimization problem is defined to find the optimal feedback gain matrix in order to access the best settling time. To solve this optimization problem three intelligent optimization methods are utilized: Particle Swarm Optimization (PSO), Artificial Bee Colony (ABC) and Bees Algorithm (BA). In addition, an optimal controller based on linear quadratic regulator (LQR) is designed and the results are compared with the optimized full state feedback controller. The results show that using the intelligent optimization methods, better settling time can be obtained and consequently better performance for the system is achieved.

Keywords: Linear Quadratic Regulator, Particle Swarm Optimization, Artificial Bee Colony, Bees Algorithm

1. Introduction

According to all UAVs classes, unmanned rotorcraft and in particular unmanned helicopters have advantages of both taking-off and landing vertically and also, they don’t require a runway and are able to hover and fly in low altitudes.

Helicopters are nonlinear coupled multiple input multiple output (MIMO) systems that include challenging design issues. For helicopters, the throttle, the collective pitch, the cyclic pitch and anti-torques pedals are the control variables. The throttle holds the rotational velocity of the main rotor constant. The collective pitch controls the magnitude of the main rotor thrust. The cyclic pitch controls the lateral and longitudinal tilting of the main rotor disk [1].

The modern control theory is more complicated compared to classical methods. It is because of using all state variables instead of using state variables considered as output. Increased complexity of the system and easy accessibility to large-scale computers, modern control theory- a new approach to analysis and to design the complex control systems - were developed. This new approach is based on the concept of state [2].

While the conventional PD or PID controllers have been shown to be insufficient to control nonlinear systems, methods based on the linear quadratic regulator (LQR) have been proven to be very efficient and are relatively simple ways used even for a highly nonlinear system like a helicopter. Also, applying optimal control allows the designer to establish a tradeoff between the controller effort and variation of states.

Besides, considering the design of the controller as an optimization problem, a variety of optimization methods can be used to solve the problem. In this paper, controller of a single input multiple output (SIMO) helicopter model is designed based on minimizing the settling times of all the state variables. Several robust optimization methods with random methodology including particle swarm optimization (PSO), artificial bee colony (ABC), bees algorithm (BA), are utilized to solve the optimization problem.

2. Helicopter Equations

The term Unmanned Aerial Vehicles (UAVs) is used to describe unpiloted flight. The applicability of UAVs is predominant in the execution of potentially dangerous flight missions or in cases where the
Small size of the vehicle restricts the presence of a pilot [1]. Helicopter is a vehicle which rotates freely and translates in all six degrees of freedom (i.e. rolling, pitching, yawing, surging, swaying and heaving).

The rigid body equations are defined in the fixed framework. Three differential equations describing the helicopter translational motions in the body framework are derived as follow

\[
\begin{align*}
\frac{du}{dt} &= b \frac{f_x}{m} + b \frac{v}{r} - b \frac{w}{q} \\
\frac{dv}{dt} &= b \frac{f_y}{m} - b \frac{u}{r} + b \frac{w}{p} \\
\frac{dw}{dt} &= b \frac{f_z}{m} + b \frac{u}{q} - b \frac{v}{p}
\end{align*}
\]

Equation (1)

Where \( u, v \) and \( w \) are translational velocities parameters and \( q, r \) and \( p \) are rotational velocities act in the \( x, y \) and \( z \) direction respectively. Similarly, the following three ordinary differential equations describing the helicopter rotational motions are derived (Note that, these equations do not depend on the reference frame).

\[
\begin{align*}
\frac{dp}{dt} &= (I_{yy} - I_{zz}) q r + L \\
\frac{dq}{dt} &= -(I_{xx} - I_{zz}) p r - M \\
\frac{dr}{dt} &= (I_{xx} - I_{yy}) q p + N
\end{align*}
\]

Equation (2)

In which L, M and N are moments components respectively defined in the \( x, y \) and \( z \) directions. I denote the inertia matrix, consider that in this case non diagonal members of inertia matrix are zero and diagonal members can be calculated by Eq. (3).

\[
\begin{align*}
I_{xx} &= \sum (y_m^2 + z_m^2) dm \\
I_{yy} &= \sum (x_m^2 + z_m^2) dm \\
I_{zz} &= \sum (x_m^2 + y_m^2) dm
\end{align*}
\]

Equation (3)

Euler angles rates are defined in Eq. (4)
\[
\begin{align*}
\phi &= p + \sin(\phi) \tan(\theta) \cdot q + \cos(\phi) \tan(\theta) \cdot r \\
\theta &= \cos(\phi) \cdot q - \sin(\phi) \cdot r \\
\psi &= \frac{\sin(\phi) \cdot q + \cos(\phi)}{\cos(\theta)} \cdot r
\end{align*}
\] (4)

Where \( \phi \) is the roll angle, \( \theta \) is the pitch angle and \( \psi \) is the yaw angle. Because these equations are derived in body fixed frame, it’s necessary to define a rotational matrix that can bring all equations in earth framework (Newtonian framework), so a rotational matrix is defined as follow to change this coordinates.

\[
R(\Theta) = \begin{bmatrix}
\cos \theta \cos \psi & \sin \theta \sin \phi \cos \psi - \cos \phi \sin \psi & \sin \theta \cos \phi \cos \psi + \sin \phi \sin \psi \\
\cos \theta \sin \psi & \sin \theta \sin \phi \sin \psi - \cos \phi \cos \psi & \sin \theta \cos \phi \sin \psi - \sin \phi \cos \psi \\
-\sin \theta & \sin \phi \cos \theta & \cos \phi \cos \theta
\end{bmatrix}
\] (5)

3. Full State Feedback Control

The reason for considering linear quadratic optimal control is that LQR designs are robust with respect to fairly large plant variations [3]. A brief summary is presented here, more details can be found in [4]. Consider the state space model of a linear time invariant (LTI) system as follow

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\] (6)

Where \( A \) is system or dynamic matrix, \( A \in \mathbb{R}^{n \times n} \), \( B \) is the input matrix, \( B \in \mathbb{R}^{n \times r} \), \( C \) is the output matrix, \( C \in \mathbb{R}^{m \times n} \). In order to establish linear feedback around the system above, a linear feedback law can be implemented which can be written as Eq. (7).

\[
u(t) = -Kx(t) + r(t)
\] (7)

Where \( K \) is the feedback matrix of dimension \( n \times m \), \( r(t) \) is the reference input vector to the system. As all of the states are measured, the resulting feedback system is called a full state feedback system [4]. An equation for the closed loop system is derived as follow

\[
\dot{x}(t) = (A - BK)x(t) + Br(t)
\] (8)

This system is stable if and only if the system matrix, \( A-BK \), has all its eigenvalues in the left half plane. Hence, selecting the \( K \) matrix which can makes the system stable is most important part of the controller designing in this case. The closed loop system with full state feedback is shown in Fig. 2.

**Figure 2.** Closed loop system with full state feedback
4. Optimal Control

Considering Eqs. (6) and (8), the performance index is selected as Eq. (9).

\[ I_0 = \int_0^\infty (x^T(t)Qx(t) + u^T(t)Ru(t))dt \]  

(9)

In which, \( Q \) and \( R \) are the weighting matrices for states and inputs respectively. \( Q \) must be positive semi-definite and \( R \) positive definite. Assume a full state feedback control law of the form:

\[ u(t) = -Gx(t) \]  

(10)

Considering this assumption that the system is controllable and observable, then the optimal feedback matrix \( G \) that minimize the performance index \( (I_0) \) is given by

\[ G = R^{-1}B^T S \]  

(11)

Where the matrix \( S \) is the solution of the Riccati equation and after solving Eq. (12), matrix \( S \) will be used to find feedback gain matrix \( G \).

\[ A^T S + SA + Q - SBR^{-1}B^T = 0 \]  

(12)

Finally the optimal performance index will be calculated using \( S \).

\[ J^* = \frac{1}{2} X_0^T S X_0 \]  

(13)

5. Optimization Algorithms

In this section, the utilized intelligent optimization methods are explained briefly.

5.1. Particle Swarm Optimization

Particle swarm optimization (PSO), introduced by Kennedy and Eberhart [5], is an evolutionary optimization algorithm inspired by social behaviors like fish schooling or birds flocking. The members of the swarm communicate with each other and obtain the information about the good positions. Then, they adjust their own position and velocity based on this information. The best previous position of each member and the position of the best member among all the swarm are used to find the new velocities (i.e. the rate of the position change) and positions as follows:

\[
\{x\}_k + 1 = \{x\}_k + \{v\}_k + 1 \\
\{v\}_k + 1 = C_1 \{v\}_k + C_2 R_1 \{L\}_k - \{x\}_k + C_3 R_2 \{G\}_k - \{x\}_k \\
\]

Where \( \{x\}_k \) and \( \{v\}_k \) are the displacement and velocity vectors at the \( k^{th} \) iteration, respectively. \( \{L\}_k \) and \( \{G\}_k \) are the best local and global positions that have been found so far. \( C_1 \) is the inertia coefficient and \( C_2 \) and \( C_3 \) are the acceleration coefficients. Besides, to add some randomness to the velocity vector, random numbers \( R_1 \) and \( R_2 \) are multiplied to the acceleration coefficients. The selection of \( C_1, C_2 \) and \( C_3 \) is problem-dependent, however, doing some trials, the inertia coefficient was set to 0.2 and the values of the acceleration coefficients were set to 1. This procedure iterates until whether the desired fitness is obtained or the maximum number of iterations is exceeded.
5.2. Bees Algorithm

Imitating the interesting foraging behaviors of honeybees, Pham et al. [6] proposed bees' algorithm (BA) to solve the optimization problems. In BA methodology, \( N \) random solutions are constructed, in which, \( N_1 \) solutions, with higher fitness values, are considered as the best solutions. Among the best solutions, \( N_2 \) solutions with greater fitness values are chosen as the elite ones. To reach better solutions, neighborhood searches around the best and elite solutions are performed. \( n_1 \) and \( n_2 \) are the number of exploitations around the best and elite solutions, respectively, while \( n_2 \) is greater than \( n_1 \). Besides, the remaining \( N-N_1 \) solutions are chosen randomly. Note that, the random solution, \( x_{\text{rand}} \), is calculated by:

\[
x_{\text{rand}} = x_{\text{min}} + \alpha \left( x_{\text{max}} - x_{\text{min}} \right)
\]

Where \( \alpha \) is a random vector which its elements are between 0 and 1. \( x_{\text{min}} \) and \( x_{\text{max}} \) are the lower and upper bounds for the solution vector, respectively. The neighborhood searching around each element of the solution vector (i.e. \( x_i \)) is performed using:

\[
x_{p_i} = x_i - r + 2\alpha \cdot r
\]

Where \( x_{p_i} \) is the \( i \)th element of the new solution vector obtained from a neighborhood search around \( x_i \) with the radius equal to \( r \). The iteration continues until the stopping criterion is met.

5.3. Artificial Bee Colony

Karaboga [7] proposed artificial bee colony (ABC) algorithm is to solve the optimization problems. Like BA, ABC imitates the natural foraging behavior of honey bees. The swarm includes three groups of bees including employed bees, onlookers and scouts. The first half of the swarm consists of the employed bees and the second half includes the onlookers. For every food source, there is only one employed bee. Note that, the movement of the employed bees and onlookers is performed by:

\[
x_i(t+1) = \theta_j(t) + \phi(\theta_j(t) - \theta_j(t))
\]

Where \( x_i \) is the new position of the bee, \( t \) is the iteration number, \( \theta_k \) is the randomly chosen employed bee, \( j \) is the dimension of the solution and \( \phi \) is a series of random variables in the range \([-1,1] \). The employed bee whose food source has been exhausted by the bees becomes a scout. The movement of the scouts is also determined by

\[
\theta_j = \theta_{j,\text{min}} + r(\theta_{j,\text{max}} - \theta_{j,\text{min}})
\]

in which \( r \) is a random number between 0 to 1. This procedure continues until the stopping criterion is met.

6. Proposed Approach

The optimization target is to find the best solution of the problem. In this paper, we are searching for the best selection of feedback gain matrix \( K \). So it's necessary to define an objective function that makes the closed loop system as fast as possible and also stable. An objective function for a time response of system is the minimum time that the response reaches to absolute error of 0.02.
Settling Time $= \min \left[ \frac{y(t) - y_d(t)}{y_d(t)} \right] \leq 0.02$ (19)

Where $y_d$ is the desired value of output. Because the system is single input multiple output (SIMO) model, objective function should be defined as follow to include all outputs.

Objective Function $= w_1 ST_1 + w_2 ST_2 + w_3 ST_3 + w_4 ST_4$ (20)

In which, $w_i$, $i = 1 \ldots 4$ are weighting factors that explain the importance of settling time related to each output. In this case, these parameters are set to 1 because all outputs have equal importance. Designing parameters for this paper are feedback gain matrix with dimension of $n \times m$.

The number of scout bees using in optimization with BA is 50, number of best selected patches is 3, number of elite selected patches is 1, number of recruited bees around best selected patches is 4, and patch radius for neighborhood search is selected 0.02.

The number of particles used in PSO is 40; $C_1$ and $C_2$ are both selected 1. Finally, the number of colony size is selected 40 for ABC.

7. Simulation Results

For a small unmanned helicopter, an ideal hovering state can be regarded as the desired output which belonged to both the posture and the speed of small unmanned helicopter is zero. In the minor perturbation conditions, it is believed that there is no tilt movement, yaw movement and lateral movement in the longitudinal motion [8]. After linearization, the system matrix and the input matrix can be found as follow:

$$A = \begin{bmatrix} 0.4625 & -24.73 & 0 & -9.81 \\ 0 & -6.8027 & -1 & 0 \\ 0 & 235.18 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 28.57 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ (21)

How to choose accurate weighting matrices is an important factor in designing LQR controller. The larger is $Q$, the more rapid is adjust speed and the larger is $R$, the smaller is the required control energy. In an actual design, we need to consider adjusting speed and control function [8]. Let $Q$ and $R$ be selected as Eq. (22), which shows the weighting matrices selected by using GA [8]. Outputs 1 to 4 are Euler angle $\theta$, translational velocity $v$, translational velocity $u$, Euler angle $\phi$ respectively.

$$Q = diag [1, 1, 1, 1], \quad R = [1], \quad K = [-1.2353 \ 4.0579 \ 0.9270 \ 4.9631]$$ (22)

![Optimal control results](image)
Figure 3 shows the simulation results for selected weighting matrices. As it can be seen, the settling time is near 2 sec for output 3 and the steady state error is -1 for output 1. In this case, weighting matrices selected by GA put the helicopter in hover after 2 second, while the steady state error for Euler angle $\theta$ makes the helicopter to originate in undesirable direction. Hence, reducing this steady state error can make the helicopter to perform more satisfied. Moreover, translational velocity $u$ reached a pick of 0.8 and followed by a down trend, then leveled off after 2 sec. $v, \phi$ parameters did not experienced wide variation and saturated soon which was expected because of longitudinal movement of helicopter.

Hereafter, the optimization methods are used to derive the optimum feedback gain matrix. Hence, weighting matrices are set to 1 in Eq. (20). In the first step, PSO is used to solve the optimization problem. These results are shown in Fig. 4.

In this case, all parameters settled after 0.57 second after 20000 times calculation of cost function. Furthermore, the steady state error is decreased by 50% which help the helicopter to originate better than previous case. However, Maximum overshoot is increased for translational velocity $u$; But overshoot percentage become dangerous for stability, when this amount causes for Euler angles not for translational velocities. Therefore, it can be seen that decreasing in settling time and steady state error outweighs increasing in overshoot just for one component. Moreover, overshoot and settling time are inversely related which means that when one increase, another one decrease.

Figure 5 shows the results of ABC. In this case, the results are very similar to the PSO based optimization. The most important difference between these two cases is the settling time value which is 0.14 second more than PSO. Maximum overshoot for translational velocity $u$ is just up of PSO. In comparison with GA, both settling time and steady state error are decreased which means better
performance for helicopter than GA; But this can be seen that settling time is increased in comparison with PSO. Consequently, ABC is better than GA, but weaker than PSO.

![BA Optimization results](image)

**Figure 6.** BA based optimization

Finally, applying BA to control system; Simulation results for BA are illustrated in Fig. 6. As it can be seen obviously, maximum overshoot for translational velocity $u$ is peaked at 2.8 which is decrease compared to ABC and PSO, but is more than LQR method yet. In this case steady state error which is related to Euler angle $\theta$ is decreased to 0.53; this value is minimal among all methods. The settling time for components is reached at 0.57 sec which is as much as PSO. Hence, BA has better settling time than LQR, GA and equal with PSO. Also, Maximum overshoot for Euler angle $\theta$ is decreased compared to GA and PSO, but is more than LQR. The steady state error has minimum value between all other methods.

![Settling time versus function evaluation for optimizations methods](image)

**Figure 7.** Settling time versus function evaluation for optimizations methods

Figure 7 shows the process of approach to the best answer for PSO, ABC and BA. Population size for all algorithms is set to 50 and maximum number for iteration is set to 400. PSO is reached to the best solution after about 1000 function evaluations and is the most rapid method in aspect of convergence. BA converged to the best solution after about 13000 function evaluations; however, after about 8000 times calculation of the cost, it was approached to a value which was much closed to the best cost. ABC reached its best cost after about 6000 function evaluations.

Results obtained from used algorithms are summarized in Table 1. Also, Table 2 includes the corresponding feedback gain matrices for each algorithm.

### Table 1. Comparison of obtained values for LQR, PSO, ABC and BA

<table>
<thead>
<tr>
<th></th>
<th>LQR</th>
<th>PSO</th>
<th>ABC</th>
<th>BA</th>
</tr>
</thead>
<tbody>
<tr>
<td>Speed of convergence</td>
<td>-</td>
<td>1000 iteration</td>
<td>8000 iteration</td>
<td>13000 iteration</td>
</tr>
<tr>
<td>Settling Time</td>
<td>2 sec</td>
<td>0.57 sec</td>
<td>0.71 sec</td>
<td>0.57 sec</td>
</tr>
<tr>
<td>Steady State Error</td>
<td>-1</td>
<td>-0.6025</td>
<td>-0.7016</td>
<td>-0.5341</td>
</tr>
<tr>
<td>Maximum Overshoot</td>
<td>0.8868</td>
<td>3.6561</td>
<td>4.0168</td>
<td>2.8354</td>
</tr>
</tbody>
</table>
According to Table 1, the speed of PSO is much more compared to other algorithms. The best settling time is related to PSO and BA simultaneously. The steady state error which has been calculated for Euler angle $\theta$ is minimal for BA in comparison with other algorithms. Maximum overshoot has been evaluated for translational velocity $u$, and has the minimum value which has been reached by LQR.

Table 2. Feedback gain matrices obtained by LQR, PSO, ABC and BA

<table>
<thead>
<tr>
<th>Method</th>
<th>Gain Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>LQR</td>
<td>$K_{LQR} = [-1.2353 \ 4.0579 \ 0.9270 \ 4.9631]$</td>
</tr>
<tr>
<td>PSO</td>
<td>$K_{PSO} = [-1.7948 \ 1.6519 \ -0.0077 \ 2.8632]$</td>
</tr>
<tr>
<td>ABC</td>
<td>$K_{ABC} = [-1.5416 \ 1.5801 \ 0 \ 2.4638]$</td>
</tr>
<tr>
<td>BA</td>
<td>$K_{BA} = [-2.0389 \ 2.1412 \ 0.0243 \ 3.5302]$</td>
</tr>
</tbody>
</table>

To investigate the stability, real part of eigenvalues for modified system matrix should be considered. These values are summarized in Table 3.

Table 3. Closed loop eigenvalues obtained by LQR, PSO, ABC and BA

<table>
<thead>
<tr>
<th>Method</th>
<th>Eigenvalues</th>
</tr>
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</table>

As it can be seen, PSO and BA make the system more stable compared to other algorithms. In comparison between PSO and BA, it can be concluded that except for output 1, rest of eigenvalues are approximately similar, while, output 1 has more negative eigenvalue when BA is used which provide more stability.

8. Conclusion

In this paper, the problem of finding the best feedback gain matrix in order to control a model helicopter in longitudinal movement has been considered as an optimization problem. Hence, in the first step, LQR method was applied to the model and then the problem of finding the best feedback gain was converted to an optimization problem. To solve this problem three intelligent optimization are utilized: Particle Swarm Optimization (PSO), Artificial Bee colony (ABC), Bees Algorithm (BA). Then, the results related to each algorithm were compared. It was concluded that BA is the best one in this case because the settling time that considered as objective function has the best value and the steady state error as well. However, PSO has satisfying performance and related obtained values for this method are much closed to BA. It should be noticed that if neglect the steady state error, PSO is the best one because of its speed to converge the best solution. Consequently, PSO, ABC and BA have better performance than GA and LQR.

9. References