Upper Bound for p-Value of the Test of Multivariate Normal Ordered Mean Vectors Against all Alternatives

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To cite this article: Abouzar Bazyari & Rahim Chinipardaz (2013): Upper Bound for p-Value of the Test of Multivariate Normal Ordered Mean Vectors Against all Alternatives, Communications in Statistics - Theory and Methods, 42:10, 1748-1758

To link to this article: http://dx.doi.org/10.1080/03610926.2011.595871

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Upper Bound for p-Value of the Test of Multivariate Normal Ordered Mean Vectors Against all Alternatives

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Consider the problem of testing the isotonic of several p-variate normal mean vectors against all alternatives. It is difficult to compute the exact p-value for this problem of testing with the classical method when the covariance matrices are completely unknown. In the present paper, a test statistic is proposed for this problem of testing. A reformulation of the test statistic is given based on the orthogonal projections on the closed convex cones and then the upper bound for p-value of the test statistic is computed.

\textbf{Keywords} Closed convex cone; Likelihood ratio test; Multivariate isotonic regression; Multivariate normal distribution; Orthogonal projection.

\textbf{Mathematics Subject Classification} Primary 62F30; Secondary 62F03; 62H15.

1. Introduction

Suppose that $X_{1}, X_{2}, \ldots, X_{m}$ are random variables from a p-variate normal distribution with unknown mean vector $\mu_i$ and non-singular covariance matrix $\Sigma_i$, $N_p(\mu_i, \Sigma_i), i = 1, 2, \ldots, k$. Consider the likelihood ratio test for testing

$$H_0 : \mu_1 \leq \mu_2 \leq \cdots \leq \mu_k,$$

against all alternatives, where $\mu_i \leq \mu_j$ means that all the elements of $\mu_j - \mu_i$ are non-negative.

Let $H_1$ be the hypothesis placing no restriction on $\mu$'s. Then it is desired to test the hypothesis $H_0$ against the alternative hypothesis $H_1 - H_0$.

Such tests may be used in some fields. Their applications can be found in clinical trails design to test superiority of a combination therapy (Laska and Meisner, 1989; Sarka et al., 1995). More examples may be found in the case of ordered treatment
means or the testing in which a treatment is better than control when the responses are ordinal in a more general setting than normality.

The tests like these started by Bartholomew (1959a) who derived a likelihood ratio test for homogeneity of \( k \) univariate normal means, i.e., \( H^* : \mu_1 = \mu_2 = \cdots = \mu_k \), against ordered alternatives. The most well-known and extensively studied approach is the likelihood ratio method. He derived the test statistics as well as their null distributions under the assumptions that the variances are known and unknown. Bartholomew (1959b) studied the problem of testing the homogeneity of \( k \) univariate normal means against two-sided ordered hypothesis, i.e., the means in the alternative either increase or decrease simultaneously as \( i \) increases (but not both). Moreover, he obtained the test criterion for the problem of testing, deriving a good approximation to its distribution in the general case, for \( k = 5 \). The problems with ordered parameters have been studied to some extents by Bartholomew (1961a,b); Chacko (1963); Shorack (1967), and Kudo and Yao (1982). Much of the development and theory on this subject was assembled in Barlow et al. (1972); Robertson et al. (1988), and Silvapulle and Sen (2005). The null distribution of the likelihood ratio test (LRT) statistic depends heavily on both the specific form of the order restriction and the sample sizes. For the case of equal sample sizes, the null distributions are well tabulated for the simple order case, \( \mu_1 \leq \mu_2 \leq \cdots \leq \mu_k \) with at least one strict inequality, and simple tree order case, \( \mu_1 \leq [\mu_2, \ldots, \mu_k] \) with at least one strict inequality, where \([\ldots]\) indicates that the means inside the bracket are unconstrained (Barlow et al., 1972; Robertson et al., 1988). For unequal sample sizes, approximations of the null distributions under these constraints are available (Chase, 1974; Siskind, 1976; Robertson and Wright, 1983, 1985; Wright and Tran, 1985). Although the null distribution is not readily available for most other orderings.

Kudo (1963) considered a \( p \)-dimensional normal distribution with unknown mean \( \mu = (\mu_1, \mu_2, \ldots, \mu_p) \) and known covariance matrix \( \Sigma \). The problem of testing was \( H'_0 : \mu = 0 \) against the restricted alternative \( H'_1 : \mu_i \geq 0, (i = 1, \ldots, p) \), where the inequality is strict for at least one value of \( i \). He obtained the statistic based on the likelihood ratio criterion and discussed its existence and geometric nature and also gave a scheme for its computation. Perlman (1969) studied this problem assuming that \( \Sigma \) is completely unknown. He derived the test statistic based on the likelihood ratio method and computed its exact distribution under the null hypothesis. Robertson and Wegman (1978) obtained the likelihood ratio test statistic for testing the isotonic of several univariate normal means against all alternative hypotheses. They calculated its exact critical values at different significance levels for some of the normal distributions and simulated the power by Monte Carlo experiment. Also they considered the test of trend for an exponential class of distributions. The problem of testing the homogeneity of several multivariate normal means against the unrestricted alternative hypothesis was given by Anderson (1984). Sasabuchi et al. (1983) extended Bartholomew (1959a) problem to multivariate normal mean vectors with known covariance matrices. They obtained the likelihood ratio test statistic, proposed an iterative algorithm for computing the bivariate isotonic regression and studied the convergence of this algorithm. Kulatunga and Sasabuchi (1984) derived the null distribution of this test statistic. Kulatunga et al. (1990) proposed some test procedures when the covariance matrices are not diagonal, and studied them by simulation. Sasabuchi et al. (1992) generalized the iterative algorithm to multivariate isotonic regression. Sasabuchi et al. (1998) made some power comparisons by simulation in the bivariate case and showed that under the ordered hypothesis, Sasabuchi et al.
(1983) test is more powerful than the usual chi-square test. Fernando and Kulatunga (2007) proposed a FORTRAN program for the computation of multivariate isotonic regression and studied the convergence of the algorithm when the dimension is greater than or equal to five through Monte Carlo simulation.

Sasabuchi et al. (2003) considered this problem of testing in case that the covariance matrices are common but unknown. He proposed a test statistic, studied its upper tail probability under the null hypothesis, and estimated its critical values. Sasabuchi (2007) then gave some more powerful tests than Sasabuchi et al. (2003).

In this article, we consider the problem of testing

\[ H_0 : \mu_1 \leq \mu_2 \leq \cdots \leq \mu_k, \]

against all alternatives on the mean vectors. In fact, this problem is a multivariate normal extension of Robertson and Wegman (1978).

The arrangement of this article is as follows. In Sec. 2, we give the definition of the multivariate isotonic regression and then the test statistic is obtained using the likelihood ratio method when the covariance matrices are known. Also, a test statistic is proposed when the covariance matrices are completely unknown and common. In Sec. 3, we give some preliminary definitions and then formulate the obtained test statistic for the case of unknown covariance matrices based on the orthogonal projections on the closed convex cones. In Sec. 4, the upper bound for the \( p \)-value of the test statistic is computed. Finally, concluding remarks are given in Sec. 5.

2. Definition and Likelihood Ratio Test Statistic

In this section, the definition of the multivariate isotonic regression is given and also the test statistic is computed based on the likelihood ratio criterion for testing \( H_0 \) against \( H_1 \) and \( H_0 \).

Definition 2.1 (Sasabuchi et al., 1983). Given \( p \)-variate real vectors \( X_1, X_2, \ldots, X_k \) and \( p \times p \) positive definite matrices \( \Sigma_1, \Sigma_2, \ldots, \Sigma_k \), a \( p \times k \) real matrix \( (\hat{\mu}_1, \hat{\mu}_2, \ldots, \hat{\mu}_k) \) is said to be the multivariate isotonic regression (MIR) of \( X_1, X_2, \ldots, X_k \) with weights \( \Sigma_1^{-1}, \Sigma_2^{-1}, \ldots, \Sigma_k^{-1} \) if \( (\hat{\mu}_1 \leq \hat{\mu}_2 \leq \cdots \leq \hat{\mu}_k) \) and \( (\hat{\mu}_1, \hat{\mu}_2, \ldots, \hat{\mu}_k) \) satisfies

\[
\min_{\mu_1 \leq \mu_2 \leq \cdots \leq \mu_k} \sum_{i=1}^{k} (X_i - \mu_i) \Sigma_i^{-1} (X_i - \mu_i) = \sum_{i=1}^{k} (X_i - \hat{\mu}_i) \Sigma_i^{-1} (X_i - \hat{\mu}_i),
\]

where \( \hat{\mu}_i \)'s can be computed with iterative algorithm proposed by Sasabuchi et al. (1983).

In fact, this definition includes the definition given by Barlow et al. (1972) for univariate variables.

Now, for testing \( H_0 \) against \( H_1 = H_0 \) based on the likelihood ratio method, we have

\[
\lambda = \frac{\sup_{H_0} L(\mu)}{\sup_{H_1} L(\mu)} = \frac{\sup_{H_0} \prod_{i=1}^{k} \frac{1}{(2\pi)^{p/2} |\Sigma_i|^{-1/2}} \exp \left\{ -\frac{1}{2} \sum_{j=1}^{n_i} (X_{ij} - \mu_i) \Sigma_i^{-1} (X_{ij} - \mu_i) \right\}}{\sup_{H_1} \prod_{i=1}^{k} \frac{1}{(2\pi)^{p/2} |\Sigma_i|^{-1/2}} \exp \left\{ -\frac{1}{2} \sum_{j=1}^{n_i} (X_{ij} - \hat{\mu}_i) \Sigma_i^{-1} (X_{ij} - \hat{\mu}_i) \right\}}.
\]

A likelihood ratio test rejects \( H_0 \) for small values of \( \lambda \).


**Theorem 2.1.** Suppose that $\Sigma_i$ is known. The likelihood ratio test for $H_0$ against $H_1$ 

$-2 \ln \lambda = \sum_{i=1}^{k} n_i (\hat{\mu}_i - \bar{X}_i)' \Sigma_i^{-1} (\hat{\mu}_i - \bar{X}_i),$

where $\bar{X}_i$ is the maximum likelihood estimate of $\mu_i$ under the alternative hypothesis and 

$(\hat{\mu}_1, \hat{\mu}_2, \ldots, \hat{\mu}_k)$ is the multivariate isotonic regression of $\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_k$ with weights $n_1 \Sigma_1^{-1}, n_2 \Sigma_2^{-1}, \ldots, n_k \Sigma_k^{-1}$ and $\bar{X}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} x_{ij}, i = 1, 2, \ldots, k.$

**Proof.** With some manipulations we have that

$\lambda = \exp \left\{ -\frac{1}{2} \left[ \sum_{i=1}^{k} \sum_{j=1}^{n_i} (X_{ij} - \hat{\mu}_i)' \Sigma_i^{-1} (X_{ij} - \hat{\mu}_i) - \sum_{i=1}^{k} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)' \Sigma_i^{-1} (X_{ij} - \bar{X}_i) \right] \right\},$

where $\bar{X}_i$ and $\mu_i$ are defined as above. Then,

$-2 \ln \lambda = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (X_{ij} - \hat{\mu}_i)' \Sigma_i^{-1} (X_{ij} - \hat{\mu}_i) - \sum_{i=1}^{k} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)' \Sigma_i^{-1} (X_{ij} - \bar{X}_i) \right] \right\},$

where $\bar{X}_i$ and $\mu_i$ are defined as above. Then,

$2 \sum_{j=1}^{n_i} (X_{ij} - \hat{\mu}_i)' \Sigma_i^{-1} (\hat{\mu}_i - \bar{X}_i) = 2 \left[ \sum_{j=1}^{n_i} X_{ij}' \Sigma_i^{-1} \hat{\mu}_i - \sum_{j=1}^{n_i} X_{ij}' \Sigma_i^{-1} \bar{X}_i - \sum_{j=1}^{n_i} \bar{X}_i \Sigma_i^{-1} \hat{\mu}_i + \sum_{j=1}^{n_i} \bar{X}_i \Sigma_i^{-1} \bar{X}_i \right]$

Finally, we have the following statistic:

$-2 \ln \lambda = \sum_{i=1}^{k} n_i (\hat{\mu}_i - \bar{X}_i)' \Sigma_i^{-1} (\hat{\mu}_i - \bar{X}_i).$

**2.1. Test Statistic when The Covariance Matrices are Unknown**

In this subsection, we suppose that the covariance matrices are unknown, common for $k$ populations and they are the same under $H_0$ and $H_1$. Therefore, we consider the problem of testing $H_0$ against $H_1$ with mentioned conditions on covariance matrices. Then the likelihood function is

$L(\mu) = \prod_{j=1}^{k} \frac{1}{(2\pi)^{p/2} |\Sigma|^{-1/2}} \exp \left\{ -\frac{1}{2} \sum_{j=1}^{n_i} (X_{ij} - \mu_i)' \Sigma^{-1} (X_{ij} - \mu_i) \right\}$

$= c |\Sigma|^{-p/2} \exp \left\{ -\frac{1}{2} \left[ \sum_{j=1}^{n_i} (X_{ij} - \mu_i)' \Sigma^{-1} (X_{ij} - \mu_i) \right] \right\},$

where $c$ is a positive constant which is independent of $\mu_i$ and $\Sigma$. 
The likelihood ratio test statistic is
\[
\lambda' = \frac{\sup_{\mu \in H_0} L(\mu)}{\sup_{\mu \in H_1 - H_0} L(\mu)},
\]
and since \( \sup \) is not dependent on \( \mu \) and \( \sup_{\mu \in H_1 - H_0} \), so we have
\[
\lambda' = \frac{\sup_{\Sigma} \sup_{\mu \in H_0} L(\mu)}{\sup_{\Sigma} \sup_{\mu \in H_1 - H_0} L(\mu)},
\]
where \( \sup_{\mu \in H_0} \) denotes the supremum for the parameters \( \mu_1, \ldots, \mu_k \) under \( H_0 \) and \( \sup_{\Sigma} \) denotes the supremum for all the \( p \times p \) positive definite matrices.

Therefore, we have
\[
\lambda' = \frac{\sup_{\Sigma} \sup_{\mu \in H_0} |\Sigma|^{-k/2} \exp \left\{ -\frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{n_i} (X_{ij} - \overline{X}_j)^\prime \Sigma^{-1} (X_{ij} - \overline{X}_j) \right\}}{\sup_{\Sigma} \sup_{\mu \in H_1 - H_0} |\Sigma|^{-k/2} \exp \left\{ -\frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{n_i} (\hat{X}_{ij} - \hat{\mu}_i)^\prime \hat{\Sigma}_0^{-1} (\hat{X}_{ij} - \hat{\mu}_i) \right\}}.
\]

where \( \hat{\Sigma}_0 \) and \( \hat{\Sigma}_1 \) are the estimators of the unknown matrix \( \Sigma \) under the hypothesis \( H_0 \) and \( H_1 - H_0 \) respectively.

So, we can write
\[
-2 \ln \lambda' = \left[ k \ln \hat{\Sigma}_0 + \sum_{i=1}^{k} \sum_{j=1}^{n_i} (X_{ij} - \overline{X}_j)^\prime \hat{\Sigma}_0^{-1} (X_{ij} - \overline{X}_j) \right]
- \left[ k \ln \hat{\Sigma}_1 + \sum_{i=1}^{k} \sum_{j=1}^{n_i} (\hat{X}_{ij} - \hat{\mu}_i)^\prime \hat{\Sigma}_1^{-1} (\hat{X}_{ij} - \hat{\mu}_i) \right]
= \left[ k \ln \hat{\Sigma}_0 + \text{tr} \left\{ \sum_{i=1}^{k} \sum_{j=1}^{n_i} \hat{\Sigma}_0^{-1} (X_{ij} - \overline{X}_j)(X_{ij} - \overline{X}_j)^\prime \right\} \right]
- \left[ k \ln \hat{\Sigma}_1 + \text{tr} \left\{ \sum_{i=1}^{k} \sum_{j=1}^{n_i} \hat{\Sigma}_1^{-1} (\hat{X}_{ij} - \hat{\mu}_i)(\hat{X}_{ij} - \hat{\mu}_i)^\prime \right\} \right],
\]
where the symbol \( \text{tr} \) denotes the trace of matrix. Now, using Lemma 3.2.2 of Anderson (1984) in the way similar to that of Anderson (1984, Sec. 8.8), in order to get the likelihood ratio test statistic for our problem we need to minimize the determinant
\[
| S + \sum_{i=1}^{k} n_i (\mu_1 - \overline{X}_i)(\mu_1 - \overline{X}_i)^\prime |
\]
under the order hypothesis \( \mu_1 \leq \mu_2 \leq \cdots \leq \mu_k \). Where \( S = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (X_{ij} - \overline{X}_j)(X_{ij} - \overline{X}_j)^\prime \) is distributed with the Wishart distribution \( W_p(N - k, \Sigma) \) and \( N = n_1 + \cdots + n_k \).
On the other hand, we have
\[
\left| S + \sum_{i=1}^{k} n_i (\mu_i - \overline{X}_i)(\mu_i - \overline{X}_i)' \right|
\]
\[= |S| \cdot \left| I_p + S^{-1/2} \sum_{i=1}^{k} n_i (\mu_i - \overline{X}_i)(\mu_i - \overline{X}_i)' S^{-1/2} \right|. \]

So, we need to minimize the term
\[
\left| I_p + S^{-1/2} \sum_{i=1}^{k} n_i (\mu_i - \overline{X}_i)(\mu_i - \overline{X}_i)' S^{-1/2} \right|. \]

Also, according to the following equation:
\[
|I_p + \varepsilon A| = \prod_{i=1}^{p} (1 + \varepsilon \lambda_i) = 1 + \sum_{i=1}^{p} \varepsilon \lambda_i + O(\varepsilon^2) = 1 + \text{tr}(\varepsilon A) + O(\varepsilon^2)
\]

Then our problem is to minimize the following term:
\[
1 + \text{tr} \left[ S^{-1/2} \sum_{i=1}^{k} n_i (\mu_i - \overline{X}_i)(\mu_i - \overline{X}_i)' S^{-1/2} \right]
\]
\[= 1 + \sum_{i=1}^{k} n_i (\mu_i - \overline{X}_i)' S^{-1}(\mu_i - \overline{X}_i), \]

Under the order hypothesis \( \mu_1 \leq \mu_2 \leq \cdots \leq \mu_k \). Where \( \hat{\mu}_1, \hat{\mu}_2, \ldots, \hat{\mu}_k \) is said to be the multivariate isotonic regression of \( \overline{X}_1, \overline{X}_2, \ldots, \overline{X}_k \) with weights \( n_1 S^{-1}, n_2 S^{-1}, \ldots, n_k S^{-1} \).

So, for testing \( H_0 \) against \( H_1 - H_0 \), with unknown covariance matrices we use the following test statistic:
\[
T = \sum_{i=1}^{k} n_i (\hat{\mu}_i - \overline{X}_i)' S^{-1}(\hat{\mu}_i - \overline{X}_i).
\]

A likelihood ratio test rejects \( H_0 \) for large values of this statistic.

3. Structure of the Test Statistic based on The Closed Convex Cones

**Definition 3.1.** Let \( x = (x_1, x_2, \ldots, x_k)' \) and \( y = (y_1, y_2, \ldots, y_k)' \) be the \( p \)-dimensional in Euclidean space \( R^p \). Then for any \( p \times p \) positive definite matrix \( A \), their inner product in \( R^{nk} \) Euclidean space is defined by
\[
\langle x, y \rangle_A = \sum_{i=1}^{k} n_i x_i' A^{-1} y_i
\]
\[= (x_1', x_2', \ldots, x_k') \begin{pmatrix} n_1 A^{-1} & 0 \\ 0 & n_2 A^{-1} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{pmatrix}.
\]
Also, a norm \( \| \cdot \|_A \) in \( R^{nk} \) is defined by
\[
\|x\|_A = (x, x)_A^{1/2}.
\]

**Definition 3.2.** Let \( C \) be a non empty subset in the Euclidean space. We call \( C \) is a convex cone if \( x, y \in C \) and \( \lambda \geq 0, \gamma \geq 0 \), then \( \lambda x + \gamma y \in C \).

Further, \( C \) is a closed convex cone if it is convex cone and closed set.

Consider the closed convex cone
\[
C_0 = \left\{ \mu = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_k \end{pmatrix} \mid \mu_1 \leq \cdots \leq \mu_k, \ \mu_i \in R^p, \ i = 1, 2, \ldots, k \right\},
\]
and \( L \) be the closed convex cone which there no restriction on the mean vectors.

Now, we need to find the orthogonal projection of parameter \( \mu_i \) on \( C_0 \) and \( L \).

Using the notation given by Sasabuchi et al. (2003), if for any \( x \in R^{kp}, \pi_A(x, C) \) denotes the orthogonal projection \( x \) onto \( C \), then \( \pi_A(x, C) \) is the point which minimizes \( \|x - z\|_A \) for any \( z \in C \).

We can write the test statistic based on the orthogonal projections
\[
T = T(\mu) = \sum_{i=1}^k n_i (\hat{\mu}_i - \bar{X}_i)' S^{-1} (\hat{\mu}_i - \bar{X}_i)
\]
\[
= \sum_{i=1}^k n_i \left[ \pi_S(X_i, C_0) - \pi_S(X_i, L) \right]' S^{-1} \left[ \pi_S(X_i, C_0) - \pi_S(X_i, L) \right]
\]
\[
= \|\pi_S(X, C_0) - \pi_S(X, L)\|_S^2
\]
\[
= \langle \pi_S(X, C_0) - \pi_S(X, L), \pi_S(X, C_0) - \pi_S(X, L) \rangle_S.
\]

Define the random vector \( Z_i = X_i - \mu_i \). It is clear that the vector \( Z_i \) is independent of \( \mu_i \) and is distributed with \( N_p(0, \Sigma) \). With \( t \) being the observed value of \( T(\mu) \), the \( p \)-value of the test is
\[
p-value = \sup_{\mu \in C_0} \sup_{\Sigma > 0} P(T(\mu) > t)
\]
\[
= \sup_{\Sigma > 0} P(T(\mu = 0) > t),
\]
where \( \sup_{\Sigma} \) denotes the supremum for all the \( p \times p \) positive definite matrices.

**4. Upper Bound for \( p \)-value**

Let for any \( k, L_k \) be a closed convex cone which for any \( i \) and \( j \), there is no restriction on the \( k \) th row of two mean vectors \( \mu_i \) and \( \mu_j \). Then it is entirely clear that \( C_0 \subset L = \bigcap_{i=1}^p L_i \subset L_k \). Now, consider the following statistic
\[
T_k = T_k(\mu) = \|\pi_S(X, C_0) - \pi_S(X, L_k)\|_S^2.
\]
Lemma 4.1. Let $M$ be a positive definite matrix. Then:
(a) $T = \|\pi_{MMSM}(MX, MC_0) - \pi_{MMSM}(MX, ML)\|_{MMSM}^2$;
(b) $T_k = \|\pi_{MMSM}(MX, MC_0) - \pi_{MMSM}(MX, ML_k)\|_{MMSM}^2$;
(c) $T_k \geq T$.

Proof. (a) In accordance with the defined inner product, for the positive definite matrix $M$, we have
\[
\langle MA, MB \rangle_{MMSM} = \sum_{i=1}^{k} n_i A_i^T M M^{-1} S^{-1} M^{-1} MB_i
\]
\[
= \sum_{i=1}^{k} n_i A_i^T S^{-1} B_i
\]
\[
= \langle A, B \rangle_S.
\]
and hence
\[
T = \|\pi_S(X, C_0) - \pi_S(X, L)\|_S^2
\]
\[
= \|M \pi_{MMSM}(X, C_0) - M \pi_{MMSM}(X, L)\|_{MMSM}^2.
\]
On the other hand, it is clear that for any closed convex cone $C$, $M \pi_S(X, C) = \pi_S(MX, MC)$. Thus, we result that
\[
T = \|\pi_{MMSM}(MX, MC_0) - \pi_{MMSM}(MX, ML)\|_{MMSM}^2.
\]
The proof of part b is similar to the proof of part a.
To prove the part c, first we show that
\[
T_k = T + \|\pi_S(X, L) - \pi_S(X, L_k)\|_S^2 + 2u,
\]
where $u = \langle \pi_S(X, C_0) - \pi_S(X, L), \pi_S(X, L) - \pi_S(X, L_k) \rangle_S$.
Suppose that $\theta_j$ be the estimator of the parameter on the closed convex cone $L_k$, then the second term of the formula (1) is
\[
\sum_{i=1}^{k} n_i (\hat{\mu}_i - \bar{X}_i)' S^{-1} (\hat{\mu}_i - \bar{X}_i) + \sum_{i=1}^{k} n_i (\bar{X}_i - \theta_j)' S^{-1} (\bar{X}_i - \theta_j)
\]
\[
+ 2 \sum_{i=1}^{k} n_i (\hat{\mu}_i - \bar{X}_i)' S^{-1} (\bar{X}_i - \theta_j)
\]
\[
= \sum_{i=1}^{k} n_i \hat{\mu}_i S^{-1} \hat{\mu}_i + \sum_{i=1}^{k} n_i \hat{\mu}_i S^{-1} \bar{X}_i - \sum_{i=1}^{k} n_i \bar{X}_i S^{-1} \hat{\mu}_i + \sum_{i=1}^{k} n_i \bar{X}_i S^{-1} \bar{X}_i
\]
\[
+ \sum_{i=1}^{k} n_i \bar{X}_i S^{-1} \bar{X}_i - \sum_{i=1}^{k} n_i \bar{X}_i S^{-1} \theta_j - \sum_{i=1}^{k} n_i \theta_j S^{-1} \bar{X}_i + \sum_{i=1}^{k} n_i \theta_j S^{-1} \theta_j
\]
\[
+ 2 \sum_{i=1}^{k} n_i \hat{\mu}_i S^{-1} \bar{X}_i - 2 \sum_{i=1}^{k} n_i \hat{\mu}_i S^{-1} \theta_j - 2 \sum_{i=1}^{k} n_i \bar{X}_i S^{-1} \bar{X}_i + 2 \sum_{i=1}^{k} n_i \bar{X}_i S^{-1} \theta_j
\]
Therefore, we get
\[
\sum_{i=1}^{k} n_i (\hat{\mu}_i - \bar{\theta}) S^{-1} (\hat{\theta}_i - \bar{\theta}) = \| \pi_S(X, C_0) - \pi_S(X, L_k) \|^2_S = T_k.
\]

On the other hand, since
\[
u = \langle \pi_S(X, C_0) - \pi_S(X, L), \pi_S(X, L) - \pi_S(X, L_k) \rangle_S
\]
\[
= \sum_{i=1}^{k} n_i (\hat{\mu}_i - \bar{X}_i) S^{-1} (\bar{X}_i - \bar{\theta}) .
\]
Then by formula 2.2 given by Sasabuchi et al. (1983), we get that
\[
\sum_{i=1}^{k} n_i (\hat{\mu}_i - \bar{X}_i) S^{-1} \bar{X}_i = 0.
\]
Therefore,
\[
u = - \sum_{i=1}^{k} n_i (\hat{\mu}_i - \bar{X}_i) S^{-1} \bar{\theta}_i
\]
\[
= \langle \pi_S(X, C_0) - \pi_S(X, L), -\pi(X, L_k) \rangle_S
\]
\[
= \langle \pi_S(X, C_0) - \bar{X}_i, -\pi_S(X, L_k) \rangle_S .
\]

Since for any closed convex cone \( C \), \( \pi(X, C) \in C \) and by Lemma 1.1 of Zarantonello (1971), for \( A \in \mathbb{R}^{p \times k} \) and \( B \in C \),
\[
\langle A - \pi(A, C), B \rangle \leq 0.
\]
Therefore, we get
\[
u = \langle \pi_S(X, C_0) - \bar{X}_i, -\pi_S(X, L_k) \rangle_S \geq 0.
\]
But according to (1), we result that \( T_k \geq T \).
Suppose that \( \mu = 0 \) and define \( Z = MX \). Then,
\[
T_k(Z) = \| \pi_S(Z, MC_0) - \pi_S(Z, ML_k) \|^2_S
\]
\[
= \| \pi_{MSM}(MX, MC_0) - \pi_{MSM}(MX, ML_k) \|^2_{MSM} .
\]
With comparing (2) with the part b of the given lemma, it is clear that \( T_k = T_k(Z) \).
But since \( T_k \geq T \), then \( T_k(Z) \geq T \). Also \( E(Z) = 0 \), \( \Sigma_Z = I \) and \( Z \) is distributed with \( N(0, I) \). So that
\[
P(T(0, \Sigma) > t) \leq P(T_k(0, I) > t) = P(T_k(Z) > t).
\]
For any observed value of \( t \geq 0 \), we have that
\[
\sup_{\Sigma \geq 0} P(T(0, \Sigma) > t) \leq P(T_k(0, I) > t).
\]
So, \( P(T_k(0, I) > t) \) is the upper bound for \( p \)-value of the test statistic.
5. Concluding Remark

In this article, we proposed a test statistic for the problem of testing the isotonic of several $p$-variate normal mean vectors against all alternatives when the common covariance matrices are unknown. The upper bound of $p$-value also obtained analytically. It should be noted that it would be an advantage to investigate the power behavior of our test using the Monte Carlo simulation as a future article. We hope to communicate the further results of work on these topics later.

Acknowledgment

The authors are deeply grateful to the referees for their important and valuable comments and suggestions that led to considerable improvements of this article.

References


