Introduction of a mathematical model for calculating sub-surface drains spacing using fractional derivatives

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ABSTRACT

One of the limitations of Boussinesq equation is that its parameters (e.g. hydraulic conductivity) are scale-dependent. In this work, a fractional Boussinesq equation was obtained by assuming power-law changes of flux in a control volume and using a fractional Taylor series. Unlike Boussinesq equation, due to the non-locality property of fractional derivatives, the parameters of fractional Boussinesq equation are constant and scale-invariant. The linear form of fractional Boussinesq equation was solved by using spectral representation and an analytical mathematical model was derived to calculate sub-surface drains spacing. The optimal values of parameters of mathematical model developed in this study and Glover-Dumm's model were estimated from inverse modelling. In the inverse methodology, water table data between two sub-surface drains and optimisation method of Bees algorithm were used. The accuracy of proposed model was investigated using water table data between the two sub-surface drains and compared to Glover-Dumm's model. The results indicated that the mathematical model derived in this study predicted the water table profile between two sub-surface drains more exactly than Glover-Dumm's model.

Key words: Bees algorithm, fractional Boussinesq equation, fractional derivative, sub-surface drainage

INTRODUCTION

The accurate design and installation of sub-surface drains are very important. The equations used to design sub-surface drains must have the existing conditions in the field if possible. Because steady-state drainage conditions rarely occur in nature, there are several analytical equations applied to unsteady-state drainage conditions. The most non-steady drainage equations (Dumm, 1954; Hammad, 1962; Kumar et al., 1994; Verma et al., 1998; Cooke et al., 2001; Stillman et al., 2006) are based on Boussinesq equation. One of the limitations of analytical solutions of Boussinesq equation is that the property of porous medium in the region considered must be assumed scale-independent and homogenous (Bear and Verruijt, 1992). However, the porous media are typically heterogeneous (Bear et al., 1968) and therefore, this assumption is not valid. In recent years, fractional derivatives have been used as a useful tool to solve these problems. The fractional derivative is a generalisation of an integer-order derivative in which the positive integer order is replaced with a positive fractional order (Podlubny, 1999).

One of the most important properties of fractional derivatives, in contrast to integer-order derivatives, is their property of non-locality. This property means that the value of the fractional derivative of a function at a given point depends on both the value of the function at the point itself and the value of the function across the entire spatial domain (Huang et al., 2008). As the order of the fractional derivative becomes smaller, the distal points have a greater effect on the character of the function (Schumer et al., 2001). Therefore, when a certain equation that is based on the fractional...
derivatives is estimated at a given point, the resultant solution is also affected by the distal points. This property has allowed the fractional derivatives to be a useful tool for studying phenomena with properties that are dependent on space and time (Ding et al., 2010). When the fractional derivatives are used for porous media, the order of the fractional derivative is indicative of the degree of heterogeneity; i.e. a smaller order of the fractional derivative corresponds to a greater extent of heterogeneity (Clarke et al., 2005). Moreover, the scale effects on properties within the control volume are eliminated by using fractional derivatives (Benson et al., 2000a; Pachepsky et al., 2003; Wheatcraft and Meerschaert, 2008). In this case, the hydraulic characteristics are scale-invariant and have constant values. Recently, fractional derivatives have been used in many diverse fields including hydrogeology, finance, physics, biology, electrical engineering and electro-magnetic theory. In the field of hydrogeology, the most frequent application of fractional derivatives has been in the simulation of solute transport in ground water. Benson et al. (2000a) suggested a fractional advection-dispersion equation (FADE) to simulate solute transport in heterogeneous aquifers. Subsequent studies (Pachepsky et al., 2000; Benson et al., 2000b; Benson, et al., 2001; Martinez et al. 2001; Zhou and Seliem, 2003) have indicated that this equation simulates solute transport in heterogeneous porous media more accurately than an advection-dispersion equation (ADE). In addition, studies have shown that the fractional Richards’ equation, which is a generalisation of the classic Richards’ equation, simulates a soil moisture curve more accurately than the classic Richards’ equation (Pachepsky et al., 2003). Wheatcraft and Meerschaert (2008) suggested a fractional mass conservation by assuming power-law changes of flux through the control volume and using a fractional Taylor series. In this paper, a fractional Boussinesq equation was obtained by using the methodology employed by Wheatcraft and Meerschaert (2008) to derive the fractional mass conservation. In addition, by considering initial and boundary conditions associated with sub-surface drains, a linear fractional Boussinesq equation for one-dimensional transient flow toward sub-surface drains was solved and a new analytical mathematical model for calculating sub-surface drains spacing was developed. Furthermore, the optimal values of parameters of proposed mathematical model in this study and Glover-Dumm’s model were estimated from inverse modelling. In the inverse methodology, water table data between two sub-surface drains and optimisation method of Bees algorithm (Ozbakir et al., 2010) were used. Finally, the accuracy of obtained model was investigated using water table data between the two sub-surface drains and compared to Glover-Dumm’s model.

Theory

Fractional Boussinesq equation

By considering the following assumptions, the fractional Boussinesq equation is developed:

A. The Dupuit-Forchhimer assumptions and Darcy’s law are valid.
B. The fluid (water) passing through the control volume is non-compressible.
C. The flux changes through the control volume follow a power-law function.

To develop the fractional Boussinesq equation, consider the fluid mass conservation for the control volume bounded by the horizontal impervious bottom of the aquifer and vertical surfaces at $x, x+\Delta x$, $y$ and $y+\Delta y$, as shown in Fig. 1.

The control volume is bounded by the horizontal impervious bottom of the aquifer and vertical surfaces at $x, x+\Delta x$, $y$ and $y+\Delta y$, as shown in Fig. 1.

![Fig. 1. Control volume in an unconfined aquifer.](image-url)
from below, while it is bounded by the water table from above.

The inflow component of fluid mass flux, \( M(x) \), which passes through \(-x\) face is:

\[
M(x) = \Delta y \rho q_x
\]

Where, \( q_x \) is the \( x \) component of the specific discharge (per unit width of aquifer, \( L^2/T \)) in the \( x \) direction and \( \rho \) is the fluid density (\( M/L^3 \)).

It is proved that the \( \theta \) th order fractional Taylor series of an \( \theta \) th order power-law function is identical to the function itself (Weartcraft and Meerschaert, 2008). Therefore, if it is assumed that the change of flux in the \( x \) direction follows a power-law function with order \( \alpha \) \([\alpha \in (0, 1)]\), one can compute the outflow of fluid mass flux passing through \( +x \) face using the two-term \( \theta \) order fractional Taylor series expanded about \( x \):

\[
M(x + \Delta x) = \Delta y \left[ \rho q_x + \frac{\partial^\alpha (\rho q_x)}{\partial x^\alpha} \frac{\Delta x^\alpha}{\Gamma(1+\alpha)} \right] \tag{2}
\]

Where, \( \Gamma(.) \) is the gamma function. In this study, it is used the Caputo fractional derivative. The fractional derivatives and the fractional Taylor series have been defined in the Appendix.

The net fluid mass flux in the \( x \) direction is attained by subtracting Eq. (1) from Eq. (3):

\[
M(x) - M(x + \Delta x) = -\frac{\partial^\alpha (\rho q_x)}{\partial x^\alpha} \frac{\Delta x^\alpha}{\Gamma(1+\alpha)} \Delta y \tag{3}
\]

Similarly, if it is assumed that the change of flux in the \( y \) direction is a power-law function with order \( \beta \) \([\beta \in (0, 1)]\), the net fluid mass flux through \( +y \) face will be:

\[
M(y) - M(y + \Delta y) = -\frac{\partial^\beta (\rho q_y)}{\partial y^\beta} \frac{\Delta y^\beta}{\Gamma(1+\beta)} \Delta x \tag{4}
\]

The water mass conservation equation in an unconfined aquifer (Bear, 1972):

\[
M(x) - M(x + \Delta x) + \frac{\partial^\alpha (\rho q_x)}{\partial x^\alpha} \frac{\Delta x^\alpha}{\Gamma(1+\alpha)} \Delta y + \frac{\partial^\beta (\rho q_y)}{\partial y^\beta} \frac{\Delta y^\beta}{\Gamma(1+\beta)} \Delta x = \rho \Delta x \Delta y \frac{\partial h}{\partial t} \tag{5}
\]

Where, \( N > 0 \) (positive downward) means accretion and \( N < 0 \) means evaporation or transpiration from the water table; \( S \) is specific yield (dimensionless) where it is neglected the elastic storativity because the storativity resulting from drainage from the pore space is much greater than that due to the elasticity of the water and soil matrix (Bear, 1978) and \( t \) is the time. As \( \Delta t \to 0 \), it is obtained:

\[
\begin{align*}
M(x) - M(x + \Delta x) + M(y) - M(y + \Delta y) + \rho N &= \rho \Delta x \Delta y \frac{\partial h}{\partial t} \tag{6}
\end{align*}
\]

Substituting Eqs. (3) and (4) into Eq. (5), it is obtained:

\[
\begin{align*}
-\frac{\partial^\alpha (\rho q_x)}{\partial x^\alpha} \frac{\Delta x^\alpha}{\Gamma(1+\alpha)} - \frac{\partial^\beta (\rho q_y)}{\partial y^\beta} \frac{\Delta y^\beta}{\Gamma(1+\beta)} + \rho N &= \rho S \frac{\partial h}{\partial t} \tag{7}
\end{align*}
\]

Due to the assumption of non-compressibility of water, Eq. (7) can be expressed as:

\[
-\frac{\partial^\alpha q_x}{\partial x^\alpha} \frac{\Delta x^\alpha}{\Gamma(1+\alpha)} - \frac{\partial^\beta q_y}{\partial y^\beta} \frac{\Delta y^\beta}{\Gamma(1+\beta)} + N = S \frac{\partial h}{\partial t} \tag{8}
\]

Because it is assumed that Darcy law is valid, \( q_x \) and \( q_y \) are equal to (Bear, 1972):

\[
\begin{align*}
q_x &= -K_x \frac{\partial h}{\partial x} \tag{9}
\end{align*}
\]

\[
\begin{align*}
q_y &= -K_y \frac{\partial h}{\partial y} \tag{10}
\end{align*}
\]

Where, \( K_x \) is the saturated hydraulic conductivity in the \( x \) direction (\( L/T \)), and \( K_y \) is saturated hydraulic conductivity in the \( y \) direction.

The negative sign represents that the flow of water is in the direction of decreasing head (Todd and Mays, 2005). Considering Darcy's law, Eq. (8) is expressed as:

\[
\begin{align*}
\frac{\partial^\alpha}{\partial x^\alpha} \left( K_x \frac{\partial h}{\partial x} \frac{\Delta x^\alpha}{\Gamma(1+\alpha)} \right) + \frac{\partial^\beta}{\partial y^\beta} \left( K_y \frac{\partial h}{\partial y} \frac{\Delta y^\beta}{\Gamma(1+\beta)} \right) + N &= S \frac{\partial h}{\partial t} \tag{11}
\end{align*}
\]

Because of non-locality property of the fractional derivatives, the hydraulic characteristics (e.g. hydraulic conductivity) are scale-invariant and have constant values (Benson et al., 2000a; Pachepsky et al., 2003; Wheatcraft and Meerschaert, 2008). Hence, Eq. (11) can be written as:

\[
\begin{align*}
\frac{\partial^\alpha}{\partial x^\alpha} \left( h \frac{\partial h}{\partial x} \frac{\Delta x^\alpha}{\Gamma(1+\alpha)} \right) + \frac{\partial^\beta}{\partial y^\beta} \left( h \frac{\partial h}{\partial y} \frac{\Delta y^\beta}{\Gamma(1+\beta)} \right) + N &= S \frac{\partial h}{\partial t} \tag{12}
\end{align*}
\]

Eq. (12) can be expressed as:

\[
K_x \frac{\Delta x^\alpha}{\Gamma(1+\alpha)} \frac{\partial^\alpha}{\partial x^\alpha} \frac{h \frac{\partial h}{\partial x}}{\Delta x^\alpha} + K_y \frac{\Delta y^\beta}{\Gamma(1+\beta)} \frac{\partial^\beta}{\partial y^\beta} \frac{h \frac{\partial h}{\partial y}}{\Delta y^\beta} + N = S \frac{\partial h}{\partial t}
\]
\[
\kappa_x \frac{\partial^\alpha}{\partial x^\alpha} \left( \frac{\partial h}{\partial x} \right) + \kappa_y \frac{\partial^\beta}{\partial y^\beta} \left( \frac{\partial h}{\partial y} \right) + N = S \frac{\partial h}{\partial t} \quad (13)
\]

Where, \( \kappa_x = K_x \frac{\Delta x^{\alpha-1}}{\Gamma(1+\alpha)} \) and \( \kappa_y = K_y \frac{\Delta y^{\beta-1}}{\Gamma(1+\beta)} \). In this work, \( K_x \) is named the fractional saturated hydraulic conductivity in the x direction \((L^x/T)\). Similarly, \( K_y \) is named the fractional saturated hydraulic conductivity in the y direction \((L^y/T)\).

Eq. (13) is the fractional Boussinesq equation for flow in a heterogeneous and anisotropic aquifer. Because the fractional order indicates the degree of heterogeneity (Clarke et al., 2005), in this paper, \( \alpha \) is called the heterogeneity index in x direction and \( \beta \) is called the heterogeneity index in y direction.

If the variation of \( h \) relative to the value of \( h \) is infinitesimal, one can consider that the average saturated thickness is equal to a constant value and derive a linear fractional Boussinesq equation:

\[
\kappa_x \cdot D \frac{\partial^\nu h}{\partial x^\nu} + \kappa_y \cdot D \frac{\partial^\mu h}{\partial y^\mu} + N = S \frac{\partial h}{\partial t} \quad (14)
\]

Eq. (14) is the linear fractional Boussinesq equation flow in a heterogeneous and anisotropic aquifer. Similarly, \( \nu \) is called the heterogeneity index in x direction and \( \mu \) is called the heterogeneity index in y direction.

Where, \( D \) is the average saturated thickness \((L)\); \( \nu = 1+\alpha \) and \( \nu \Sigma (1, 2) \); \( \mu = 1+\beta \) and \( \mu \Sigma (1, 2) \).

Eq. 29 can be written as:

\[
C_x \cdot \frac{\partial^\nu h}{\partial x^\nu} + C_y \cdot \frac{\partial^\mu h}{\partial y^\mu} + N = \frac{\partial h}{\partial t} \quad (15)
\]

Where, \( C_x = \frac{\kappa_x \cdot D}{S} \) is the fractional hydraulic dispersion coefficient in the x direction \((L^\nu/T)\) and \( C_y = \frac{\kappa_y \cdot D}{S} \) is the fractional hydraulic dispersion coefficient in the y direction \((L^\mu/T)\).

**Mathematical Model for Calculating Sub-surface Drains Spacing**

To develop a mathematical model for calculating sub-surface drains spacing, the linear fractional Boussinesq equation was solved for one-dimensional transient toward sub-surface drains. To achieve this aim, the following assumptions were considered:

A. Flow toward sub-surface drains is horizontal. To account for radial flow near the drain, actual depth of the impermeable layer below drain \( D \) is replaced with Hooghoudt’s equivalent depth \( d_e \).

B. Unsaturated flow above the water table is neglected.

C. The initial water table is flat.

D. Recharge occurs instantaneously and the water table rises suddenly.

E. There is a horizontal impermeable layer at a constant depth below drains.

F. The sub-surface drains have equal spacing and lie in a parallel manner above the impermeable layer.

For one-dimensional transient flow, the linear fractional Boussinesq equation is:

\[
\frac{\partial h(x,t)}{\partial t} = C_x \frac{\partial^\nu h(x,t)}{\partial x^\nu} \quad (16)
\]

Where, \( C_x \) is equal to \( \frac{\kappa_x \cdot D}{s} \). As indicated in Fig. 2, the initial and boundary conditions for

---

**Fig. 2. Considered sub-surface drains in this study.**

Solving the one-dimensional linear fractional Boussinesq equation in sub-surface drains are:

\[
h(x,0) = h_0 \quad 0 < x < L \quad (17)
\]

\[
h(0,t) = 0 \quad (18)
\]

\[
h(L,t) = 0 \quad (19)
\]

Eq. 16 is solved using spectral representation (Ilic et al., 2005, 2006). To this end, the eigen values are \( \lambda_n^* = \frac{\pi n}{L} \) for \( n = 1, 2, 3, \ldots \) and the corresponding eigen functions are nonzero constant multiples of \( e^{i\lambda_n^* x} \). Next, \( h(x, t) \) is given by:

---
The problem To ordinary differential equations is obtained is substituting into Eq. (16), it is obtained:

\[ \sum_{n=1}^{\infty} \frac{d b_n(t)}{dt} \sin \left( \frac{n \pi}{L} x \right) = -C_v \sum_{n=1}^{\infty} b_n(t) \lambda_n^2 \sin \left( \frac{n \pi}{L} x \right) \]  

Substituting Eqs. (20) and (21) into Eq. (16), it is obtained:

\[ \sum_{n=1}^{\infty} \frac{d b_n(t)}{dt} \sin \left( \frac{n \pi}{L} x \right) = -C_v \sum_{n=1}^{\infty} b_n(t) \lambda_n^2 \sin \left( \frac{n \pi}{L} x \right) \]  

The problem for \( b_n(t) \) becomes a system of ordinary differential equations:

\[ \frac{d b_n(t)}{dt} + C_v \lambda_n^2 b_n(t) = 0 \]

The general solution of Eq. (23) is:

\[ b_n(t) = b_n(0) \exp \left( -C_v \lambda_n^2 t \right) \]  

To obtain \( b_n(0) \), it is used the initial condition (Eq. 17):

\[ h(x,0) = h_0 = \sum_{n=1}^{\infty} b_n(0) \sin \left( \frac{n \pi}{L} x \right) \]  

From Eq. (25), it is concluded that:

\[ b_n(0) = \frac{2}{L} \int_0^L h_0 \sin \left( \frac{n \pi}{L} \xi \right) d \xi = \frac{2 h_0}{n \pi} \left( 1 - \cos(n \pi) \right) \]  

Substituting \( \lambda_n = \sqrt{\frac{n \pi}{L}} \) and \( b_n(0) = \frac{2 h_0}{n \pi} \left( 1 - \cos(n \pi) \right) \) into (24), it is obtained that:

\[ b_n(t) = \frac{2 h_0}{n \pi} \left( 1 - \cos(n \pi) \right) \exp \left( -C_v \frac{n \pi}{L} t \right) \]  

Substituting \( b_n(t) \) into Eq. (20), the solution of Eq. (16) is obtained:

\[ h(x,t) = \frac{2 h_0}{\pi} \sum_{n=1}^{\infty} \left( 1 - \cos(n \pi) \right) \exp \left( -C_v \frac{n \pi}{L} t \right) \sin \left( \frac{n \pi}{L} x \right) \]  

Eq. (28) is a new equation for calculating profile of the water table between two sub-surface drains under unsteady state conditions. This equation is applicable both for heterogeneous soil and for homogeneous soil which is a special type of heterogeneous soil. When \( v \to 2 \), Eq. (28) reduces to the Glover-Dumm's model which was developed by assuming homogeneity of soil (Ritzema, 1994).

**MATERIALS AND METHODS**

**Performance of Proposed Mathematical Model**

To evaluate the performance of proposed mathematical model in this study, water table profile of the sub-surface drains installed on a pilot in Abadan (30°1’49.8” N latitude and 48°29’52.77” E longitude) was measured. The area of this pilot is 12 ha. The soil of the site was silty clay in texture. The sub-surface drains were spaced 30 m apart at a depth of 1.2 m. The water table was monitored using observation wells. The data of hydraulic head for a period of 10 days at an interval of 24 h were measured. The optimal values of parameters of mathematical model developed in this study (\( C_v \) and \( v \)) and Glover-Dumm's model (\( C_y \)) were estimated from inverse modelling. To achieve this aim, water table data between two sub-surface drains at times \( t=1, 3, 5 \) and 7 days after the beginning of the drainage and the optimisation method of Bees algorithm were used. Then, the water table profiles between two sub-surface drains were predicted at times \( t=4 \) and 10 days after the beginning of the drainage using estimated parameters for two models and the accuracy of each model was determined. The relative performance of each model was represented by a goodness-of-fit index that it was calculated by the sum of the squared deviations between the observed and predicted values of water table height (Singh et al., 1996).

**RESULTS AND DISCUSSION**

**Properties of Fractional Boussinesq Equation**

The fractional Boussinesq equation is a generalisation of the Boussinesq equation used for all soil types (homogeneous or heterogeneous and isotropic or anisotropic soils). The terms \( \frac{\partial^v}{\partial (\partial x)^v} \) and \( \frac{\partial^v}{\partial (\partial y)^v} \) in Eq. (12) introduce the scale effects in the x and y directions, respectively. Low values of \( a(\alpha(0, 1)) \) and \( \beta(\beta \varepsilon \)
(0, 1)) indicate that the scale effect on the hydraulic properties is large and the porous medium is very heterogeneous. By contrast, when the values of $\alpha$ and $\beta$ decrease to 1, the components of $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$ also decrease to 1, and the scale effect is eliminated. In this case, the porous medium is homogeneous and anisotropic. Equal values of $\alpha$ and $\beta$ indicate that the heterogeneity is equal in the $x$ and $y$ directions. When $\alpha=\beta=1$ and $(K_x=K_y=K)$, the non-linear Boussinesq equation in a homogeneous and isotropic porous medium is given. Similarly, the equal values of $v$ and $\mu$ in the linear fractional Boussinesq equation indicate an equal degree of heterogeneity in the $x$ and $y$ directions. When $v=\mu=2$, the porous medium is homogeneous and anisotropic.

Verification of the Proposed Mathematical Model

The estimated parameters of two mathematical models for the mentioned case study are reported in Table 1. The results of evaluation of the prediction accuracy of water table profile by two mathematical models at times $t=4$ and 10 days after the beginning of the drainage are shown in Figs. 3 and 4, respectively.

<table>
<thead>
<tr>
<th>Site</th>
<th>Mathematical model</th>
<th>$V$</th>
<th>$C_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abadan</td>
<td>Proposed model</td>
<td>1.26</td>
<td>$6.55 \times 10^6$ $m^{1.36}/s$</td>
</tr>
<tr>
<td></td>
<td>Glover-Dumm's model</td>
<td>-</td>
<td>$3.47 \times 10^5$ $m^{1.5}/s$</td>
</tr>
</tbody>
</table>

Therefore, the mathematical model developed in this study predicts the water table profile between two sub-surface drains more exactly than Glover-Dumm’s model. It is thought that the assumption of homogeneity of soil that considered for derivation of the Glover-Dumm’s model might be one of the most important reasons that the proposed mathematical model in this study had a better relative performance. Obviously, this is not a correct assumption and hydraulic characteristics of soil are typically scale-dependent. However, to derive the proposed mathematical model in this study, the assumption of homogeneity of soil is not necessary. In this case, due to the non-locality property of the fractional derivatives, the hydraulic characteristics are scale-invariant and have constant values.

CONCLUSION

A fractional Boussinesq equation was obtained by assuming power-law changes of flux in a control volume and using a fractional Taylor series. Because of the non-locality property of fractional derivatives, the parameters of fractional Boussinesq equation
are constant and scale-invariant. An analytical mathematical model for calculating subsurface drains spacing was developed by solving a linear fractional Boussinesq equation for one-dimensional transient flow toward sub-surface drains. The proposed mathematical model predicts the water table profile between two sub-surface drains more exactly than Glover-Dumm's model.

**Appendix**

There are several definitions of a fractional derivative of order $\alpha>0$. The two most widely known and applied definitions are the Riemann-Liouville and Caputo derivatives. The Caputo space-fractional derivative of order $\alpha$ for $a<x<b$ is defined as follows (Al-rabtah et al., 2010):

$$\left(D_{a}^{\alpha}f\right)(x,t) = \frac{1}{\Gamma(m-\alpha)} \int_{a}^{t} \frac{f^{(m)}(y,t)}{(t-y)^{\alpha}} dy \quad m-1<\alpha \leq m \quad (A1)$$

Where, $m$ is the smallest integer greater than $\alpha$. The Caputo fractional derivative is commonly used to model real-world phenomena because it allows traditional initial and boundary conditions to be included in the formulation of the problem (Al-rabtah et al., 2010). The fractional Taylor series of $f(x,t)$ for $x$ at time $t$ in the Caputo sense is defined as (Odibat and Shawagfeh, 2007):

$$f(t+\Delta t)=f(t)+\frac{\partial f(t)}{\partial t}\Delta t^1+\frac{\Delta^2 f(t)}{2!}\Delta t^2+...+\frac{\partial^m f(t)}{\partial t^m}\Delta t^m \quad (A2)$$

Where, $0<\alpha<1$

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