An analytical approach for radial-forward extrusion process

R. Ebrahimi a,⁎, M. Reihanian b, M.M. Moshksar a

aDepartment of Materials Science and Engineering, School of Engineering, Shiraz University, Shiraz, Iran
bDepartment of Materials Science and Engineering, Faculty of Engineering, Shahid Chamran University, Ahvaz, Iran

1. Introduction

Extrusion is a bulk metal forming operation in which the cross section area of a billet is reduced and changed into a certain shape by forcing it through a die. Depending on the direction of the punch motion and the direction of the material flow, extrusion process is classified into three basic types; named as forward (direct) extrusion, backward (indirect) extrusion and radial extrusion. In addition to these basic extrusion operations, there are some combined extrusion processes in which two (or more) basic extrusion processes occur simultaneously. Some examples of these combined processes are: backward-forward extrusion [1–3], backward extrusion forging [4,5], double-backward extrusion [6], multi-step extrusion [7], radial-backward extrusion [8] and radial-forward extrusion [9,10]. The most important characteristic of these processes is the production of the complex shaped components in a single action operation. Regarding their industrial importance, the analysis of the forming characteristics of the combine extrusion processes, may be an important task, because it can help the design of tooling. The analysis of the combined extrusion processes have been extensively carried out by many researchers using both analytical and finite element methods [1–7]. However, little attempts have been made to analyze the radial-forward and radial-backward extrusion. Only a rigid-plastic finite element method has been used to analyze the radial-forward and radial-backward extrusion processes [8–10]. In these papers the effect of the geometrical parameters and the friction factor on the material flow and tooling loads were investigated.

2. Method

2.1. Analysis of the radial-forward extrusion

Fig. 1 shows the die geometry and deformation model considered for radial-forward extrusion. During extrusion process, a billet is forced up through a container until it reaches the bottom surface of a stationary mandrel. By increasing the force, the material flows radially in the gap between the bottom surface of the mandrel and the die surface. The radial flow continues until the material reaches the vertical wall of the die. Further increasing of the force, causes the material extruded directly in the vertical gap between the die wall and the mandrel. The volume considered for strain values and upper-bound analysis was divided into five regions. In regions I and V the material moves rigidly with the velocities of and respectively. Regions II, III and IV are deformation regions in which the material undergoes plastic deformation. These regions are separated by the velocity discontinuity surfaces OC, BC, DE and EF. In addition to these surfaces, there are some frictional surfaces between the material surfaces, mandrel surfaces and die walls. These are HC, AB, BE, CD, DF, CJ, and FI surfaces.

2.2. Strain analysis

2.2.1. Components of the velocity fields

By using a cylindrical coordinates (r, θ, z) and considering the symmetry of the deformation, the tangential components of the velocity field, , will be zero in the deformation regions II, III and IV. That is:

⁎ Corresponding author. Tel./fax: +98 711 2307293.
E-mail address: ebrahimj@shirazu.ac.ir (R. Ebrahimi).
Nomenclature

\[ dV \] differential volume element of deformation zone
\[ dS \] differential surface element of frictional or velocity discontinuity surfaces
\[ F_{\text{ext}} \] extrusion force
\[ k \] shear yield stress
\[ l \] container length
\[ l_d \] die contact length
\[ l_m \] mandrel contact length
\[ m \] constant friction factor
\[ O \] center of the cylindrical coordinates
\[ P_{\text{ext}} \] extrusion pressure
\[ r \] radial distance from center of the cylindrical coordinates
\[ R \] radius of the container (radius of the billet)
\[ R_i \] radius of the mandrel (inner radius of the tube)
\[ R_o \] radius of the die (outer radius of the tube)
\[ t \] deformation time
\[ T \] the gap between the bottom surface of the mandrel and the die surface (flange thickness)
\[ V \] volume of the deformation zone
\[ V_e \] entrance velocity
\[ V_f \] exit velocity
\[ w_i \] internal power
\[ w_p \] power dissipated on the velocity discontinuity surfaces
\[ w_{\text{total}} \] total power dissipated on the all frictional and velocity discontinuity surfaces as well as internal power
\[ W_{\text{ext}} \] external power
\[ z \] axial distance from center of the cylindrical coordinates
\[ \Delta z \] punch displacement at the deformation time
\[ r, \theta, z \] axis of the cylindrical coordinates
\[ \varepsilon_{ij} \] components of the strain rates
\[ \varepsilon \] effective strain rate
\[ \varepsilon_m \] mean value of the effective strain rate
\[ \sigma_y \] yield stress of the undeformed material
\[ \sigma_m \] mean flow stress of the material
\[ \Delta \dot{v} \] amount of the velocity discontinuity on the frictional or discontinuity surface

\[ \dot{U}_1, \dot{U}_2, \dot{U}_3 \] components of the velocity field

**Fig. 1.** Die geometry and the deformation model considered for analysis.

\[ U_{\text{in}} - U_{\text{in}} - U_{\text{out}} = 0 \]  \hspace{1cm}  (1)

The other two components of the velocity field \((\dot{U}_1, \dot{U}_2)\) in deformation regions II, III and IV can be obtained by using the law of volume constancy.

Based on the volume constancy, from Fig. 2, the amount of material which is passed through the surface OC and entered into the volume OABC, is exited through surface B'C'. Then:

\[ (\pi R^2) \dot{V}_e = (2\pi t) \dot{U}_{\text{in}} \]  \hspace{1cm}  (2)

Thus:

\[ \dot{U}_{\text{in}} = \frac{\dot{V}_e R^2}{2t} \]  \hspace{1cm}  (3)

where \( T \) is the gap between the bottom surface of the mandrel and the die surface (thickness of the flange), \( r \) is the radius distance from center coordinate O, and \( \dot{U}_{\text{in}} \) is the radial component of the velocity field in region II.

Referring to Fig. 3, the amount of material which is passed through surface O'C' and entered into the volume D'ABC, is exited through surface B'C'. Therefore:

\[ (\pi R^2) \dot{U}_{\text{in}} = 2\pi R(T - z) \dot{U}_{\text{in}} = \frac{\dot{V}_e R^2}{2t} (T - z) \]  \hspace{1cm}  (4)

which yields:

\[ U_{\text{in}} = \frac{V_e (T - z)}{t} \]  \hspace{1cm}  (5)

where \( R \) is the radius of the billet, \( z \) is the axial distance from center O and \( U_{\text{in}} \) is the axial component of the velocity field in region II.

In region III the material has only radial flow \((U_{\text{in}} = U_{\text{in}} = 0)\). It means that \( U_{\text{in}} \) is the only non-zero component of the velocity field. Regarding Fig. 4, the amount of material which is passed through the surface OC, is exited through surface D'EG.

Hence:

\[ (\pi R^2) \dot{V}_e = (2\pi t) \dot{U}_{\text{in}} \]  \hspace{1cm}  (6)

which yields:

\[ U_{\text{out}} = \frac{V_e R^2}{2t} \]  \hspace{1cm}  (7)

By referring to Fig. 5 and using the volume constancy for the volume element D'EG, yields:

\[ (2\pi t) \dot{U}_{\text{out}} = \pi (R_o^2 - r^2) \dot{V}_i \]  \hspace{1cm}  (8)

whereas:

\[ \dot{V}_i = \frac{R_o^2 - R_e^2}{R_o^2 - R_i^2} \dot{V}_i \]  \hspace{1cm}  (9)

Inserting Eq. (9) into Eq. (8) yields:

\[ U_{\text{out}} = \frac{V_e R^2 (R_o^2 - r^2)}{2t (R_o^2 - R_i^2)} \]  \hspace{1cm}  (10)
The radius of the die (outer radius of the produced tube) and rate components are expressed by:

\[ \dot{\epsilon}_U = \frac{\partial U_z}{\partial r} \]

Using the volume constancy for volume element DEFG of Fig. 6, the following expression can be obtained:

\[ \pi(r_1^2 - r_0^2)U_{z_{II}} - (2\pi r_0 z)(U_{z_{IV}})_{z_{II}} = \frac{\pi V R^2 z}{l} \]

Thus:

\[ U_{z_{II}} = \frac{V R^2 z}{(R_0^2 - R_1^2)l} \]

where \( U_{z_{II}} \) is the axial component of the velocity field in the region IV.

2.2.2. Components of the strain rate

The relationships between the components of the velocity field and the strain rate components are expressed by:

\[ \dot{\epsilon}_{ij} = \frac{1}{2} \left( \frac{\partial U_i}{\partial r_j} + \frac{\partial U_j}{\partial r_i} \right) \]

By inserting Eqs. (3) and (5) into Eq. (13), and considering \( U_{z_{II}} = 0 \), the strain rate components in region II are determined as:

\[ \dot{\epsilon}_{r r} = \frac{V}{2l}, \quad \dot{\epsilon}_{z z} = \frac{V}{2l}, \quad \dot{\epsilon}_{r z} = -\frac{V}{l} \]

Replacing Eq. (7) into Eq. (13) and considering \( U_{z_{III}} = 0 \), the components of the strain rate in region III are obtained as:

\[ \dot{\epsilon}_{r r} = \frac{V R^2}{2 \pi r^2} \]

where \( R_o \) is the radius of the mandrel (inner radius of the produced tube), \( R_i \) is the radius of the die (outer radius of the produced tube) and \( U_{z_{IV}} \) radial component of the strain rate in region IV.

Substituting Eqs. (10) and (12) into Eq. (13) and considering \( U_{z_{IV}} = 0 \), the components of the strain rate in region IV are determined as:

\[ \dot{\epsilon}_{r r} = \frac{-V R^2}{2 \pi r^2} \]

The effective strain rate, \( \dot{\varepsilon} \), in terms of the strain rate components is expressed by:

\[ \dot{\varepsilon} = \sqrt{\frac{\dot{\epsilon}_{r r}^2 + \dot{\epsilon}_{z z}^2 + \dot{\epsilon}_{r z}^2}{3}} \]

Inserting Eqs. (14)–(16) into Eq. (17) yields:

\[ \dot{\varepsilon}_{r r} = \frac{V}{l} \]

where \( \dot{\varepsilon}_{r r} \) and \( \dot{\varepsilon}_{z z} \) are the effective strain rates in regions II, III and IV, respectively. By referring to Eqs. (19) and (20), the effective strain rate in regions III and IV are a function of radius \( r \). Thus a mean value of the effective strain rate, \( \dot{\varepsilon}_{me} \), is considered in these regions using the equation below:

\[ \dot{\varepsilon}_{me} = \frac{\dot{\varepsilon}_{r r} + \dot{\varepsilon}_{z z} + \dot{\varepsilon}_{r z}}{3} \]

where \( V \) is the volume of the deformation region and \( dV \) is a differential volume element considered for the deformation region.
Substituting Eqs. (19) and (20) into Eq. (21) yields:

\[ \dot{\varepsilon}_{\text{region II}} = \frac{2V \cdot R^2 \ln \left( \frac{s}{s_{0}} \right)}{\sqrt{3}(R_0^2 - R^2)} \]  

(22)

and

\[ \dot{\varepsilon}_{\text{total}} = \frac{V \cdot R^2 \left( 2R_0^2 - R_0^2 \log 6 - \sqrt{R_0^2 + 3R_0^2 + R_0^2 \log \left( \frac{2R_0^2 - \sqrt{R_0^2 + 3R_0^2}}{R_0^2} \right)} \right)}{\sqrt{3}(R_0^2 - R^2)^2} \]  

(23)

where \( \dot{\varepsilon}_{\text{region II}} \) and \( \dot{\varepsilon}_{\text{total}} \) are the mean value of the effective strain rates in region III and IV, respectively.

2.2.3. Deformation Time

Referring to Fig. 1, the deformation time for removing the material from region II, \( t_0 \), is equal to the time of punch movement by a distance \( T \), thus:

\[ t_0 = \frac{T}{V} \]  

(24)

The deformation time for removing the material from region III, \( t_m \), is proportional to the punch movement, \( \Delta x_m \), which is obtained easily using the volume constancy as below:

\[ \left( \pi R^2 \right) \Delta x_m = \pi \left( R_0^2 - R^2 \right) T \]  

(25)

which yields:

\[ \Delta x_m = \frac{T \left( R_0^2 - R^2 \right)}{R^2} \]  

(26)

Thus:

\[ t_m = \frac{\Delta x_m}{V} = \frac{T \left( R_0^2 - R^2 \right)}{V R^2} \]  

(27)

The extrusion time for removing the material from region IV, \( t_w \), is proportional to the punch movement, \( \Delta x_w \), which is obtained easily using the volume constancy as below:

\[ \left( \pi R^2 \right) \Delta x_w = \pi \left( R_0^2 - R^2 \right) T \]  

(28)

which yields:

\[ \Delta x_w = \frac{T \left( R_0^2 - R^2 \right)}{R^2} \]  

(29)

Thus:

\[ t_w = \frac{\Delta x_w}{V} = \frac{T \left( R_0^2 - R^2 \right)}{V R^2} \]  

(30)

2.2.4. Effective strain

The effective strain in each region is obtained simply by multiplying the effective strain rate to the corresponding deformation time as follows:

\[ \dot{\varepsilon}_{\text{II}} = \dot{\varepsilon}_0 \cdot t_0 = 1 \]  

(31)

\[ \dot{\varepsilon}_{\text{III}} = \dot{\varepsilon}_m \cdot t_m = \frac{2}{\sqrt{3}} \ln \left( \frac{R_0}{R} \right) \]  

(32)

\[ \dot{\varepsilon}_{\text{IV}} = \dot{\varepsilon}_w \cdot t_w = \frac{2R_0^2 - R_0^2 \log 6 - \sqrt{R_0^2 + 3R_0^2 + R_0^2 \log \left( \frac{2R_0^2 - \sqrt{R_0^2 + 3R_0^2}}{R_0^2} \right)}}{\sqrt{3}(R_0^2 - R^2)^2} \]  

(33)

where \( \dot{\varepsilon}_0 \), \( \dot{\varepsilon}_m \) and \( \dot{\varepsilon}_w \) are the effective strains in regions II, III, and IV, respectively.

2.3. Upper-bound analysis

Based on the upper-bound theory, for a rigid-plastic Von-Misses material, among all kinematically admissible velocity fields, the actual one minimizes the power expressed by the equation below:

\[ J = 2k \int \frac{1}{2} \left( \dot{\varepsilon}_{\text{II}} \right)^2 + \int \dot{\varepsilon}_{\text{III}} \right)^2 + \int \dot{\varepsilon}_{\text{IV}} \right)^2 + \int T \dot{\varepsilon}_{\text{II}} \right)^2 \]  

(34)

where \( k \) is the shear yield stress, \( \dot{\varepsilon}_{\text{II}} \) the components of strain rate, \( m \) the constant friction factor, \( V \) the volume of plastic deformation zone, \( S_0 \) and \( S_0 \) the area of velocity discontinuity and frictional surfaces, \( S_0 \) the area where the tractions may occur, \( \Delta \) the amount of velocity discontinuity on the frictional and discontinuity surfaces, \( \dot{\varepsilon}_{\text{II}} \) and \( T \) the velocity and tractions applied on the \( S_0 \) respectively.

The first term in the equation above represents the power dissipated in the deformation zone. The second and third term express the power dissipated along the velocity discontinuity and frictional surfaces. The last term reflects the power due to the predetermined body tractions.

Referring to Fig. 1, the total power dissipated in the radial-forward extrusion, \( w_{\text{total}} \), is obtained by:

\[ w_{\text{total}} = w_{\text{II}} + w_{\text{III}} + w_{\text{IV}} + w_{\text{region II}} + w_{\text{region III}} + w_{\text{region IV}} + w_{\text{friction}} \]  

(35)

where \( w_{\text{II}} \), \( w_{\text{III}} \), and \( w_{\text{IV}} \) are the internal power for deformation in regions II, III and IV, respectively. \( w_{\text{region II}}, w_{\text{region III}}, \) and \( w_{\text{region IV}} \) are the power dissipated on the velocity discontinuity surfaces OC, BC, DE and EF, respectively and the rests are the power dissipated on the die walls and the mandrel surfaces with the subscript denoting the frictional surface between the material and the die and/or mandrel surface. Detailed expressions of the all dissipated powers are presented in the Appendix.

The external power of radial-forward extrusion \( w_{\text{ext}} \) is expressed by:

\[ w_{\text{ext}} = w_{\text{friction}} - \pi R^2 P_{\text{ext}} V \]  

(36)

where \( P_{\text{ext}} \) and \( P_{\text{ext}} \) are the force and pressure needed for radial-forward extrusion, respectively.

By equating the external power of Eq. (36) to the total power derived in Eq. (35), the extrusion pressure for radial-forward extrusion is obtained as:

\[ P_{\text{ext}} = \frac{2\pi m_1}{3} \left[ \frac{k R_0^2 \log 6 - \sqrt{R_0^2 + 3R_0^2 + R_0^2 \log \left( \frac{2R_0^2 - \sqrt{R_0^2 + 3R_0^2}}{R_0^2} \right)}}{R_0^2} \right] \]  

(37)

where \( m_1 \) is the strength of the undeformed material, \( m_{\text{II}}, m_{\text{III}}, \) and \( m_{\text{IV}} \) are the mean flow stress of the material in regions II, III and IV respectively, \( l \) is the container length, \( l_0 \) is the mandrel contact length and \( l_0 \) is the die contact length. Defining the mean flow stress for each region is because of considering the work hardening effect of the material.

2.4. Experimental procedures

To obtain the experimental data, a radial-forward extrusion tool-setting was designed and constructed. The material used in this work was a commercially pure Al, machined from an initial rod to the length of 40 mm and diameter of 10 mm and then annealed at 420°C for 2 h. Radial-forward extrusion process was carried out at room temperature with a constant punch speed of 0.2 mm/s. Shaving foam was used as lubricant. Fig. 7 shows the sample produced by radial-forward extrusion process. The outer and inner diameter in the tube part of the component was 22 and 20 mm, respectively. The thickness of the component in flange part was 2 mm.

The constant friction factor, \( m \), appropriate for forming process was estimated by the "Barrel Compression Test" [11]. Compression tests were carried out at the same temperature and using the same lubricant as for radial-forward extrusion process. A constant friction factor of 0.1 was obtained.

The Vickers hardness method was used for hardness testing. The hardness of the material in different parts of the sample was taken by applying a load of 1 kg for 10 s on a section parallel to the extrusion axis.

![Fig. 7. A typical product of radial-forward extrusion process.](image)
Using the compression tests, the work hardening equation of the material used in this work was obtained as below:

\[ \sigma = 146e^{0.211} \text{ (MPa)} \]  \hspace{1cm} (38)

The initial yield stress of the undeformed material, \( \sigma_y \), was obtained as 58 MPa by a tensile test. Based on the Eq. (38) the mean flow stress in regions II, III and IV is that is \( \sigma_{mII}, \sigma_{mIII} \) and \( \sigma_{mIV} \) was obtained as 119, 157 and 177 MPa, respectively.

3. Results and discussions

Using the Eqs. (31)–(33) and for \( R = 5 \text{ mm}, R_i = 10 \text{ mm} \) and \( R_o = 11 \text{ mm} \), the strain imposed in regions II, III and IV are obtained as 1, 0.8 and 1, respectively. Thus the total strain achieved in the component through radial-forward extrusion is determined as 2.8 by summing the individual effective true strains in each deformation region. This strain is very large and comparable to the strain imposed through severe plastic deformation methods [12]. For validity of the strain analysis, Vickers hardness of the material was measured in regions II, III and IV. Fig. 8 shows the variation of the hardness as a function of the strain imposed in each region. The results of the hardness of the same material processed by the equal channel angular pressing (ECAP) is also plotted as a function of the strain. The ECAP process is a severe plastic deformation method in which very large strain is imposed into the material and thus it is a promising method for producing ultrafine-grained and nanostructured materials [12]. The hardness data of the ECAP process were extracted from the results of our previous work [13]. It is seen that at the same strain level, the hardness of the material is approximately the same for two processes of ECAP and radial-forward extrusion. It is a confirmation for this strain analysis that large strain is imposed during radial-forward extrusion.

Fig. 9 shows the variation of the strain imposed through radial-forward extrusion process as a function of the initial billet diameter for a component with fixed dimensions \( (R_i = 10 \text{ mm}, R_o = 11 \text{ mm} \) and \( T = 2 \text{ mm} ) \). It is seen that by reducing the initial diameter of the billet, larger plastic strain can be achieved for a component with specified dimensions. Thus in radial-forward extrusion, it may be possible to produce a component with very high strength by reducing the initial diameter of the billet.

The theoretical load–displacement curves for different values of initial billet diameter are plotted in Fig. 10. It is seen that by reducing the initial billet diameter, the extrusion force necessary for radial-forward extrusion decreases. In general, the extrusion force decreases with reducing the initial diameter while increases due to work hardening effect, which is larger in billets with smaller initial diameter. However, in this case, the former is more significant and the extrusion force decreases by reducing the initial billet diameter.

The load–displacement curve shown in Fig. 10 can be divided into three zones. In the first zone, the punch force increases gradually with the punch displacement. This part of the curve represents the radial flow of the material through the flange part of the extrusion die. The gradual increase of the punch force is due to the increase of the contact frictional surfaces between the material and die surfaces as the punch is advanced. When the material reaches to the end of the flange part, it flows into the vertical gap between the cylindrical surface of the die and the lateral surface of the mandrel. This causes a sudden change in the slope of the load–displacement curve. After entirely filling the vertical die gap, the frictional surface area remains constant as the punch is advanced; thereby, a steady-state condition is appeared.

To verify the results of the upper-bound analysis, the theoretical load–displacement curve obtained by the analysis was compared with the experimental results in Fig. 11. Results show a good agreement between the theory and experiment. It is emphasized that a slight difference in load prediction by the theory is due to the nature of the upper-bound method which overestimates the required load. The approximate difference between the theoretical and experimental load–displacement curve in the steady-state condition is about 6%. The lacks of theoretical result at low displacements is due to initial small plastic deformation of the specimen in order to fit the die cavity in the early stages of the
Fig. 11. Comparison of the theoretical and the experimental load–displacement curve.

punch displacement. This effect was ignored in the theoretical analysis.

4. Conclusions

A new analytical method is used to determine the amount of the strain imposed in radial-forward extrusion process. The total strain achieved by radial-forward extrusion is large and comparable to the strain imposed through severe plastic deformation methods. In a component with specified dimensions, larger plastic strain and thus higher strength can be achieved by reducing the initial billet diameter. Based on the upper-bound theory, an analytical equation is derived for extrusion pressure of the radial-forward extrusion process as a function of material properties (flow stress) and design parameters. There is a good agreement between the results of the strain and upper-bound analysis with the experimental data in terms of the hardness measurements and the extrusion force. A slight difference in load prediction by the theory is due to the nature of the upper-bound method which overestimates the required load. It is concluded that radial-forward extrusion is an effective process for producing of the complex shape components with a high value of strength.

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Appendix 1. Internal power in the deformation zones

Based on the upper-bound theory, the internal power dissipated in the deformation zone, \( w_i \), is given by:

\[
\dot{w}_i = \frac{2}{\sqrt{3}} \sigma_z \int_0^L \sqrt{2} \sqrt{E} \sqrt{\dot{q}} dV
\]  

(A.1)

For three deformation regions II, III and IV we have:

\[
dV_{\text{II}} = dV_{\text{III}} = dV_{\text{IV}} = 2\pi Rd\theta
\]  

(A.2)

where \( dV_{\text{II}}, dV_{\text{III}} \) and \( dV_{\text{IV}} \) are the differential volume elements in regions II, III and IV respectively. Inserting Eqs. (14) and (A.2) into Eq. (A.1) yields:

\[
\dot{w}_i = 2\pi \sigma_m V \int_0^R r dr = \pi \sigma_m V R^2
\]  

(A.3)

where \( w_i \) is the internal power dissipated in region II. Substituting Eqs. (15) and (A.2) into Eq. (A.1) results:

\[
\dot{w}_i = \frac{2\pi \sigma_m V R^2}{\sqrt{3}} \int_0^R \frac{dr}{r} = \frac{2\pi \sigma_m V R^2}{\sqrt{3}} \ln \frac{R}{r}
\]  

(A.4)

where \( \dot{w}_i \) is the internal power dissipated in region III. Replacing Eqs. (16) and (A.2) into Eq. (A.1) gives:

\[
\dot{w}_i = \frac{2\pi \sigma_m V R^2}{\sqrt{3} \left(R_0^2 - R_1^2\right)} \int_0^R \frac{1}{r} \sqrt{r^4 + R^4} dr
\]

\[
= \frac{\pi \sigma_m V R^2}{\sqrt{3} \left(R_0^2 - R_1^2\right)} \left[2R_2^2 - R_1^2 \log \left(\sqrt{R_2^2 + \sqrt{R_1^2 + R_3^4}}\right) + R_1^2 \log \left(\frac{R_1^2 + \sqrt{R_1^2 + R_3^4}}{R_2^2}\right)\right]
\]  

(A.5)

where \( \dot{w}_i \) is the internal power dissipated in region IV.

Appendix 2. Power dissipated on the velocity discontinuity surfaces

The power dissipated on the velocity discontinuity surface is expressed by:

\[
\dot{w}_v = \int_S k |\Delta v| dS
\]  

(A.6)

For velocity discontinuity surface OC:

\[
\Delta v_{\text{OC}} = U_{2a} - U_{2b} = \frac{V}{2} - 0
\]

\[
dS_{\text{OC}} = 2\pi r dr
\]

\[
\dot{w}_{\text{OC}} = \frac{\pi \sigma_m V R^2}{\sqrt{3}} \int_0^R r dr = \frac{\pi \sigma_m V R^3}{3\sqrt{3}}
\]  

(A.9)

For velocity discontinuity surface BC:

\[
\Delta v_{\text{BC}} = U_{3a} - U_{3b} = \frac{T}{R} - \frac{T}{R} V_a - 0 = \frac{T}{R} V_a
\]

\[
dS_{\text{BC}} = 2\pi R dz
\]

\[
\dot{w}_{\text{BC}} = \frac{2\pi \sigma_m V R^2 R}{\sqrt{3}} \int_0^T \left(1 - \frac{z}{R}\right) dz = \frac{\pi \sigma_m V R^2 T}{\sqrt{3}}
\]  

(A.12)

For velocity discontinuity surface DE:

\[
\Delta v_{\text{DE}} = U_{4a} - U_{4b} = \frac{V}{2} - 0
\]

\[
dS_{\text{DE}} = 2\pi R dr
\]

\[
\dot{w}_{\text{DE}} = \frac{2\pi \sigma_m V R^2 R}{\sqrt{3}} \int_0^T dz = \frac{\pi \sigma_m V R^2 T}{\sqrt{3}}
\]  

(A.15)

For velocity discontinuity surface EF:

\[
\Delta v_{\text{EF}} = U_{5a} - U_{5b} = \frac{V}{2} - 0
\]

\[
dS_{\text{EF}} = 2\pi r dr
\]

\[
\dot{w}_{\text{EF}} = \frac{2\pi \sigma_m V R^2}{\sqrt{3} \left(R_0^2 - R_3^2\right)} \int_0^R r dr = \frac{\pi \sigma_m V R^2}{\sqrt{3} \left(R_0^2 - R_3^2\right)} \left[\frac{2R_2^2 - R_1^2 \log \left(\frac{R_1^2 + \sqrt{R_1^2 + R_3^4}}{R_2^2}\right)}{R_2^2} + \frac{R_1^2 \log \left(\frac{R_1^2 + \sqrt{R_1^2 + R_3^4}}{R_2^2}\right)}{R_1^2}\right]
\]  

(A.18)

Appendix 3. Power dissipated on the frictional surfaces

The power dissipated on the frictional surface is expressed by:

\[
\dot{w}_f = \int_s m k |\Delta v| dS
\]  

(A.19)
For frictional surface HC:

\[ \Delta V_{\text{HC}} = V_s, \]
\[ \Delta S_{\text{HC}} = 2\pi RL, \]
\[ w_{t_{\text{HC}}} = \frac{2\pi m_0 V_s R L}{\sqrt{3}}. \]

For frictional surface AB:

\[ \Delta V_{\text{AB}} = U_s = \frac{V}{2T} r, \]
\[ dS_{\text{AB}} = 2\pi r dr, \]
\[ w_{t_{\text{AB}}} = \frac{\pi m_0 V_s R^2}{3T} \int_0^R r^2 dr = \frac{\pi m_0 V_s R^2 (R_2 - R)}{3T}. \]

For frictional surface CD:

\[ \Delta V_{\text{CD}} = U_{\text{CD}} = \frac{V R^2}{2T} (R_0^2 - r^2), \]
\[ dS_{\text{CD}} = 2\pi r dr, \]
\[ w_{t_{\text{CD}}} = \frac{\pi m_0 V_s R^2}{3T} \int_0^{R_0} (R_0^2 - r^2) dr = \frac{\pi m_0 V_s R^2}{3T} \left( \frac{2R_0^3}{3} - R_2^3 R_0 + \frac{R_0^3}{3} \right). \]

For frictional surface DF:

\[ \Delta V_{\text{DF}} = U_{\text{DF}} = \frac{V R^2}{T (R_0^2 - R_2^2)} z, \]
\[ dS_{\text{DF}} = 2\pi R_0 dz, \]
\[ w_{t_{\text{DF}}} = \frac{2\pi m_0 V_s R^2}{\sqrt{3}} \int_0^T z dz = \frac{\pi m_0 V_s R^2 R_0 T}{\sqrt{3} (R_0^2 - R_2^2)}. \]

For frictional surface FI:

\[ \Delta V_{\text{FI}} = V_f, \]
\[ \Delta S_{\text{FI}} = 2\pi R_i l_i, \]
\[ w_{t_{\text{FI}}} = \frac{2\pi m_0 V_f R_i l_i}{\sqrt{3}} = \frac{2\pi m_0 V_s R^2 R_i l_i}{\sqrt{3} (R_0^2 - R_2^2)}. \]

For frictional surface EJ:

\[ \Delta V_{\text{EJ}} = V_f, \]
\[ \Delta S_{\text{EJ}} = 2\pi R_i l_i, \]
\[ w_{t_{\text{EJ}}} = \frac{2\pi m_0 V_f R_i l_i}{\sqrt{3}} = \frac{2\pi m_0 V_s R^2 R_i l_i}{\sqrt{3} (R_0^2 - R_2^2)}. \]

References