A state space model for transient flow simulation in natural gas pipelines

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1. Introduction

Natural gas transportation and distribution are commonly accomplished in many countries through the gas pipelines and networks. Due to the on-line controlling and reasons that are incidental or accidental to the operation of gas transmission pipelines or networks, transient flows do commonly arise. Thus, pipeline operations are actually transient processes and in fact, steady state operations are rare in practice. The governing equations for a transient subsonic flow analysis of natural gas in pipelines are a set of two nonlinear hyperbolic partial differential equations. Many algorithms and numerical methods such as implicit and explicit finite differences, method of characteristics, state space, natural gas, and gas pipeline have been applied by several researchers for transient flow in gas pipelines (Wylie et al., 1971; Luongo, 1986; Yow, 1971; Osiadacz, 1987; Ibraheem and Adewumi, 1996; Kiuchi, 1994). However, almost all of these conventional schemes are time consuming, especially for gas network analysis.

Some of the investigators (Wylie et al., 1971; Luongo, 1986) have neglected the inertia term in momentum equation to linearize partial differential set of equations. However, it will result in loss of accuracy. Yow (1971) introduced the concept of inertia multiplier to partially account the effect of the inertia term. Osiadacz (1987) simulated transient gas flow with isothermal assumption without neglecting any terms in momentum equation for gas networks. Kiuchi (1994) used an implicit method to analyze unsteady gas networks at isothermal conditions. Also, Dukhovnya and Adewumi (2000) and Zhou and Adewumi (1996) did flow simulation with the same assumptions and by using TVD schemes. Tentis et al. (2003) have used an adaptive method of lines to simulate the transient gas flow in pipelines. Ke and Ti (1999) analyzed isothermal transient gas flow in the pipeline networks using the electrical models for the loops and nodes. Recently, Gonzales et al. (2009) have used MATLAB-Simulink and prepared some S-functions to simulate transient flow in gas networks. At their work, two simplified models have been derived containing Crank–Nicolson algorithm and method of characteristics. Reddy et al. (2006) have proposed an efficient transient flow simulation for gas pipelines and networks using the transfer functions in Laplace domain. They derived the equivalent transfer functions for the governing equations and then, using the convolution theorem, they obtained the series form of the output in the time domain. As the direct methods are very time consuming, in the two of the newest works the numerical methods are applied (Behbahani-Nejad and Shekari, 2010). Behbahani-Nejad and Shekari (2010) proposed a reduced-order modeling approach for natural gas transient flow in pipelines. They derived the linearized form of the Euler equations and obtained the corresponding eigensystem. Then, they used a few dominant flow eigenmodes to construct an efficient reduced-order
model. Behbahani-Nejad and Bagheri (2010) prepared a MATLAB-Simulink library to simulate the transient flow in gas pipelines and networks. They derived the transfer functions of a single pipeline to develop a MATLAB-Simulink library and then extended it for a gas pipeline network simulation. In the recent years, the state space models were used for diverse applications (Margaria et al., 2001; Pearson and Kotta, 2004; Berendrecht et al., 2006; Rueda and Rodríguez, 2010; McCausland et al., 2011). However, there is a rarity in applying this model for natural gas pipelines. This model can be very efficient for simulation of transient flow in natural gas pipelines. In the latest work, Behbahani-Nejad et al. (2010) simulated the transient flow in gas pipelines and networks by the state space model. They used a state space model with known inlet gas pressure and outlet gas flow rate for simulation.

In the present study, the state space equations are employed for simulation of transient flow in gas pipelines and networks. For this purpose, the mathematical model of the transient flow in a gas pipeline is presented. Next, the flow transfer functions for different boundary conditions are derived based on the mathematical model. Finally, the state space equations are obtained using the transfer functions and are employed to analyze transient flow in a gas pipeline and a gas network.

2. Mathematical model

The set of partial differential equations describing the general one-dimensional compressible gas flow dynamics through a pipeline under isothermal conditions is obtained by applying the conservation of mass, momentum and an equation of state relating the pressure, density, and temperature. For a general pipe as shown in Fig. 1, these hyperbolic partial differential equations are (Kralik et al., 1998)

\[ \frac{\partial p}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0 \]  
\[ \frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho u^2 + P)}{\partial x} = -\frac{\rho u u_f}{2D} + \rho g \sin \alpha \]  
\[ P = \rho Z \rho g T \]  
(3)

where \( \rho \) is the gas density, \( P \) is the pressure, \( u \) is the gas axial velocity, \( g \) is the gravitational acceleration, \( \alpha \) is the pipeline inclination, \( f \) is the friction coefficient, \( Z \) is the gas compressibility factor, and \( D \) is the pipeline diameter.

The governing equations in matrix form are

\[ \frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = R \]  
(4)

where

\[ F = \begin{bmatrix} \rho u \\ \rho u^2 + P \end{bmatrix}, \quad R = \begin{bmatrix} 0 \\ -\frac{\rho u u_f}{2D} + \rho g \sin \alpha \end{bmatrix} \]  
(5)

Another form of the relations (1) and (2) versus the gas pressure and mass flow rate can be written as (Kralik et al., 1998)

\[ \frac{\partial P}{\partial t} + \frac{1}{A} \frac{\partial m}{\partial x} = 0 \]  
(6)

\[ \frac{\partial P}{\partial x} = -\frac{1}{A} \frac{\partial m}{\partial t} - \frac{\partial}{\partial x} \left( \frac{1 + kP}{AP} \right) \frac{\partial m}{\partial t} \]  
\[ - \frac{f}{2DA} \frac{P}{P_0} \frac{\partial m}{\partial t} - \frac{g}{L} \frac{\partial h}{\partial t} \frac{P}{(1 + kP)RT} \]  
(7)

where \( m \) shows the mass flow rate and \( k \) is an experimental parameter which is used to compute the compressibility factor, i.e.

\[ Z = 1 + \hat{k} P. \]  
(8)

3. Flow transfer functions

To obtain the flow transfer functions, \( P_0, T_0, A_0, \) and \( \rho_0 \) are considered as the reference values and the nonlinear partial differential equations (6) and (7) are linearized with regard to them. Moreover, these reference values are also considered to define the corresponding dimensionless variables expressed as

\[ \xi = \frac{x}{L}, \quad t' = \frac{t}{L}, \quad P' = \frac{P}{P_0}, \quad m' = \frac{m}{P_0 A_0}, \quad u' = \frac{u}{u_0} \]  
(9)

where \( u_0 \) is the average gas velocity in the pipe and is calculated as (Kralik et al., 1998)

\[ u_0 = \frac{(m_{in} + m_{out})Z_0 RT_0}{(P_{in} + P_{out})A_0}. \]  
(10)

When the governing Equations (6) and (7) are linearized and the nondimensional variables are used, with some mathematical manipulations one obtains (Kralik et al., 1998)

\[ \frac{\partial \Delta m^*}{\partial t} = -\frac{\partial \Delta P^*}{\partial t} \]  
(11)

\[ \begin{bmatrix} 1 - u^2 \end{bmatrix} \frac{\partial \Delta P^*}{\partial t} = -\frac{\partial \Delta m^*}{\partial t} + 2u^* \frac{\partial \Delta P^*}{\partial t} - \left[ \frac{f}{2} \frac{u^*}{c^2} \right] \Delta m^* \]  
\[ + \frac{gL^2 u^*}{c^2} \Delta P^* \]  
(12)

where

\[ \Delta P^* = P' - P_0^* \]  
(13)

\[ \Delta m^* = m^* - m_0^*. \]  
(14)
Since for the practical subsonic transient flows $u^* = u_0/c << 1$, one can omit $u^* \cdot u^*$ at the left hand side of (12). Taking the Laplace transform of (11) and (12), yields the following two coupled linear ordinary differential equations

\[
\begin{align*}
\frac{\partial \Delta m^*}{\partial s} &= -s \Delta P^*(s) \\
\frac{\partial \Delta P^*(s)}{\partial s} &= \left[ \left| u^* \right| \left| f^* \right| - s \right] \Delta m^*(s) + \left\{ f^* \cdot u^* - \frac{g \Delta h}{c^2} + 2u^* s \right\} \Delta P^*(s).
\end{align*}
\]

(15)

(16)

After imposing the boundary conditions, the above system of ODE can be solved. For example, if the gas pressure at the inlet and the mass flow rate at the pipe outlet are specified as functions of time, the above system of ODE results in (Kralik et al., 1998)

\[
\begin{align*}
\Delta P^*_\text{out}(s) &= e^{\gamma/2} \left( \frac{2b}{2b \cosh(b) - \gamma \sinh(b)} \Delta P^*_\text{in}(s) - \frac{2a \sinh(b)}{2b \cosh(b) - \gamma \sinh(b)} \Delta M^*_\text{out}(s) \right)
\end{align*}
\]

(17)

\[
\begin{align*}
\Delta M^*_\text{in}(s) &= \frac{2b}{2b \cosh(b) - \gamma \sinh(b)} \Delta P^*_\text{in}(s) + e^{-\gamma/2} \frac{2b \cosh(b) - \gamma \sinh(b)}{2b \cosh(b) - \gamma \sinh(b)} \Delta M^*_\text{out}(s)
\end{align*}
\]

(18)

where $\alpha$, $\beta$, $b$ and $\gamma$ are defined in Annex A. After Taylor-expansion of the hyperbolic terms in (17), the simplified transfer functions are

\[
\begin{align*}
\Delta P^*_\text{out}(s) &= \frac{F_{\text{in}} \cdot P_{\text{in}}}{F_{\text{out}} \cdot P_{\text{out}} + F_{\text{in}} \cdot P_{\text{in}} + F_{\text{out}} \cdot P_{\text{out}}} \Delta M^*_\text{in}(s) + \frac{F_{\text{out}} \cdot P_{\text{out}}}{F_{\text{out}} \cdot P_{\text{out}} + F_{\text{in}} \cdot P_{\text{in}} + F_{\text{out}} \cdot P_{\text{out}}} \Delta P^*_\text{out}(s)
\end{align*}
\]

(19)

\[
\begin{align*}
\Delta M^*_\text{out}(s) &= \frac{F_{\text{in}} \cdot P_{\text{in}}}{F_{\text{out}} \cdot P_{\text{out}} + F_{\text{in}} \cdot P_{\text{in}} + F_{\text{out}} \cdot P_{\text{out}}} \Delta P^*_\text{in}(s) + \frac{F_{\text{out}} \cdot P_{\text{out}}}{F_{\text{out}} \cdot P_{\text{out}} + F_{\text{in}} \cdot P_{\text{in}} + F_{\text{out}} \cdot P_{\text{out}}} \Delta M^*_\text{out}(s)
\end{align*}
\]

(20)

\[
\begin{align*}
F_{\text{in}} \cdot P_{\text{in}}(s) &= \frac{k_1}{1 + a_1 s} \\
F_{\text{out}} \cdot P_{\text{out}}(s) &= \frac{c_1 s}{1 + a_1 s} \\
F_{\text{out}} \cdot M_{\text{out}}(s) &= -k_2 \frac{1 + b_1 s}{1 + a_1 s} \\
F_{\text{in}} \cdot M_{\text{in}}(s) &= \frac{1}{1 + d_1 s}
\end{align*}
\]

(21)

(22)

Transfer function models are used by Reddy et al. (2006), Gonzales et al. (2009) and Behbahani-Nejad and Bagheri (2010). Although the answers by this method are good in comparison to other methods, this method cannot be applied to a complicated network. Fig. 2 shows a MATLAB-Simulink transfer function model which is used for a triangle network by Behbahani-Nejad and Bagheri (2010). It seems complicated somehow. Now imagine how complicated and bewildering it can be if this model is used for a network consisting of hundreds of pipelines. However, this problem is solved in the state space model. The number of pipelines used in the network makes no difference in simulating the state space model. The equations of each pipeline just have to be written consequently in this model. This method is explained widely in the subsequent sections.

4. State space model

When the flow transfer functions are obtained, they are used to derive the state space equations for transient analysis. The block diagram of the model shown in Fig. 3 also shows the selected variables as states of the system ($x_1$–$x_4$), two inputs, gas pressure at the inlet and mass flow rate at the outlet, two outputs, the outlet pressure and the inlet gas flow rate. Thus, the state space equations are
Outputs of the systems can be written as follows

\[ P_{\text{out}} = x_1 + \left(1 - \frac{b_1}{a_1}\right) x_3 - \frac{b_1 k_2}{a_1} M_{\text{out}} \]  

\[ M_{\text{in}} = -c_1 x_2 + x_4 - \frac{c_1}{a_1} P_{\text{in}}. \]

For other boundary conditions similar relations can be obtained, which are mentioned in Annex B.

Initial conditions that are needed to solve differential equations (26)–(29) are not known a priori and should be guessed or obtained somehow. In the present study their steady state values are obtained and set as the initial conditions. The proposed approach is now extended to simulate a gas network. A typical network which has been studied by Osiadacz (1987), Ke and Ti (1999) and Behbahani-Nejad and Bagheri (2010) is considered and simulated with the present approach. Fig. 4 shows this network schematically. The accuracy of the obtained results of the proposed simulation is discussed in the next section.

5. Results and discussions

The results of the proposed transient simulation are compared with the obtained results of Reddy et al. (2006) for a single pipeline. In order to verify the accuracy of the present state space model, a 8000 m long pipeline of 0.406 m diameter with an elevation of 1 m was considered as the test case. The pipeline transports natural gas with 0.675 specific gravity at 27 °C. The gas viscosity is \(1 \times 10^{-5}\) N s/m², while the pipeline wall roughness is 0.046 mm. At the pipeline's inlet, the gas pressure is kept constant at 6 MPa, whereas the pipe's mass flow rate at the outlet varies with a 6000-s cycle, corresponding to changes in consumer demand within 6000 s as depicted in Fig. 5.

Fig. 6 illustrates the present results of state space model for mass flow rate time changes at the pipe inlet along with the obtained
results by Reddy et al. (2006). It can be seen that both models are in
good agreement and they produce identical results.

A 72,259.5 long pipeline of 0.2 m diameter was considered as
another case test. The experimental data is available for this
problem and has been studied by Taylor et al. (1962), Zhou and
Adewumi (1996), Tentis et al. (2003) and Behbahani-Nejad and
Bagheri (2010). The pipeline transports natural gas of 0.675
specific gravity at 10 °C. The gas viscosity is $1.1831 \times 10^{-5}$ N s/m²,
while the pipeline wall roughness is 0.617 mm and isothermal
sound speed equals 367.9 m/s. At the pipeline’s inlet, the gas
pressure is kept constant at 4.205 MPa, whereas the pipe’s mass
flow rate at the outlet varies with a 24-h cycle, corresponding to
changes in consumer demand within a day as is depicted in Fig. 7.

Fig. 8 illustrates the results of the present state space model for
pressure time changes at the pipe outlet, along with those of the
others (Zhou and Adewumi, 1996; Tentis et al., 2003; Taylor et al.,
1962) and the experimental results. The results of the present
state space model are comparable with other results and behave
like nonlinear finite difference models.

A typical network as shown in Fig. 4 was considered to validate
the results of the present gas network simulation. The geometrical
data of the network is introduced in Table 1 and the gas demand at
the nodes 2 and 3 are illustrated in Fig. 9. The pressure source in the
network is node 1, which is maintained at a constant pressure of
5 MPa. The gas specific gravity is approximately 0.6, the operational
temperature is 5 °C, and the friction factor is considered to be
constant and equal to 0.003. This network is simulated by state
space model with two methods. In the first method, the trial and
error method is used. The pressure constancy in each junction node
is used as an approval for answers in each step. In the second
method, the model is simulated directly; however, the boundary
conditions which are used for simulation are one step lag. The
present simulation results are compared with those obtained by

![Fig. 4. The gas pipeline network.](image)

![Fig. 5. A 6000-s irregular flow imposed at the pipe outlet.](image)

![Fig. 6. Comparison of mass flow rate time history at the inlet.](image)

![Fig. 7. A 24-h irregular flow imposed at the pipe outlet.](image)
Osadacz (1987), Ke and Ti (1999) and Behbahani-Nejad and Bagheri (2010) in Figs. 10 and 11. The obtained results are in good agreement with the others. As it can be seen, the state space results are between the other answers. Because the exact answer is unknown, it can be concluded that the state space results have less error than other methods, or even they are in the same order of the exact answers. Table 2 shows the CPU time for simulation with the two mentioned methods. With considering that these two methods’ answers are very close and looking to the CPU times, one can easily conclude that the direct method is much more efficient than the trial and error method.

Finally, for investigating the accuracy of the state space model for different boundary conditions, the second test case in this paper was simulated by two different boundary conditions from which are used in the mentioned case. In the first case, the inlet and outlet gas mass flow rates are known. The calculated outlet pressure by this method, which is mentioned in the Section 4 is compared by the previous one and is shown in Fig. 12. In the second case, the inlet and outlet pressures are known. The calculated outlet gas mass flow rate by this method is compared by the exact one and is shown in Fig. 13. The results of the present state space models show the good accuracy of applying the different boundary conditions.

Table 1
Pipe geometrical data for the considered network.

<table>
<thead>
<tr>
<th>Gas pipe ID</th>
<th>From node</th>
<th>To node</th>
<th>Diameter (m)</th>
<th>Length (km)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>3</td>
<td>0.6</td>
<td>80</td>
</tr>
<tr>
<td>2</td>
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<td>2</td>
<td>0.6</td>
<td>90</td>
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<tr>
<td>3</td>
<td>2</td>
<td>3</td>
<td>0.6</td>
<td>100</td>
</tr>
</tbody>
</table>

Table 2

<table>
<thead>
<tr>
<th>Time (Hour)</th>
<th>Flow rate (S·m³/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
</tr>
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<td>6</td>
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</tr>
<tr>
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<td>25</td>
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<tr>
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<td>20</td>
<td>55</td>
</tr>
<tr>
<td>22</td>
<td>60</td>
</tr>
</tbody>
</table>

Fig. 8. Comparison of pressure time history at the outlet.

Fig. 9. Demands versus time for nodes 2 and 3 of the simulated network.

Fig. 10. Outlet pressure results for node 2.

Fig. 11. Outlet pressure results for node 3.
6. Conclusion

The proposed simulation can be applied to analyze the transient flow of natural gas in pipelines and networks with different boundary conditions and a sufficient accuracy. Since the proposed simulation uses the state space of the transient gas flows, it is more computationally efficient than the other finite difference methods. On the other hand, one can assemble the state space equations of all the network pipes to simulate the dynamic behavior of a gas network. Thus, the state space model can be used efficiently for more complicated networks. However, as the present simulation is based on the flow state space equations, it only can give the endpoints results, not those distributions along the pipelines. Fortunately, in plenty gas networks problems the endpoints data are usually interested.

Acknowledgment

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Annex A

In this annex, the algebraic expressions of the parameters used in (17) and (19)–(28) are presented. \( \alpha, \beta, \gamma \) and \( b \) which are used in (17) are stated as (Kralik et al., 1998)

\[
\alpha(s) = \alpha_1 + \alpha_2 s
\]

\[
\beta(s) = \beta_1 s
\]

\[
\gamma(s) = \gamma_1 + \gamma_2 s
\]

\[
b(s) = \sqrt{\gamma^2 + 4\alpha\beta}/2
\]

where

\[
\alpha_1 = |u^+|fL, \quad \alpha_2 = \frac{L}{c}, \quad \beta_1 = \frac{L}{c}, \quad \gamma_1 = \frac{fL}{2}|u^+| - \frac{g\Delta h}{c^2}.
\]

The other parameters which have been used in (19)–(28) are (Behbahani-Nejad and Bagheri, 2010)

\[
\tilde{\alpha}_1 = e^{\gamma_1}\left(\frac{1}{2}\alpha_1\beta_1\left(1 - \frac{1}{6}\gamma_1 + \frac{1}{24}\gamma_1^2 - \frac{1}{240}\gamma_1^3 + \cdots\right)\right) - \frac{\gamma_2}{2}
\]

\[
k_1 = e^{\gamma_1}
\]

\[
a_1 = \tilde{\alpha}_1 - \frac{1}{2}\gamma_2
\]

\[
k_2 = e^{\gamma_1}\alpha_1\left(1 + \frac{1}{24}\gamma_1^2 + \frac{1}{1920}\gamma_1^4\right)
\]

\[
b_1 = \frac{\alpha_2}{\alpha_1} + \frac{\alpha_1\beta_1 + \frac{1}{12}\gamma_1\gamma_2}{1 + \frac{1}{4}\gamma_1^2 + \frac{1}{320}\gamma_1^4}
\]
\[ c_1 = e^{\gamma_0} \beta_1 \left( 1 + \frac{1}{24} \gamma_2^2 + \frac{1}{192} \gamma_4^4 \right) \] (A-11)

\[ d_1 = \tilde{a}_1 + \frac{1}{2} \gamma_2. \] (A-12)

**Annex B**

In this annex, state space relations for other boundary conditions rather than known inlet gas pressure and outlet gas mass flow rate are presented.

For known inlet and outlet gas flow rates, the state space equations are:

\[ \hat{x}_1(t) = \frac{1}{c_1 d_1} M_{in} - \frac{1}{c_1 d_1} M_{out} \] (B-1)

\[ \hat{x}_2 = x_1 - \frac{1}{a_1} x_2 + \frac{1}{c_1} M_{in} - \frac{\tilde{a}_1}{c_1 d_1} M_{out} \] (B-2)

\[ \hat{x}_3 = \frac{k_1}{a_1 \tilde{a}_1 d_1} M_{in} - \frac{k_1}{a_1 \tilde{a}_1 d_1} M_{out} \] (B-3)

\[ \hat{x}_4 = x_3 - \left( \frac{1}{a_1 b_1 d_1} + \frac{1}{a_1 b_1 d_1} + \frac{1}{a_1 d_1 d_1} + \frac{1}{b_1 d_1 d_1} \right) x_6 \] (B-4)

\[ x_4 = x_3 - \left( \frac{1}{a_1 b_1 d_1} + \frac{1}{a_1 b_1 d_1} + \frac{1}{a_1 d_1 d_1} + \frac{1}{b_1 d_1 d_1} \right) x_6 \] (B-13)

\[ \hat{x}_5 = x_4 - \frac{1}{a_1} x_6 + \frac{1}{a_1} \frac{1}{d_1} x_6 + \frac{1}{a_1} \frac{1}{d_1} x_6 + \frac{1}{a_1} \frac{1}{d_1} x_6 \] (B-5)

\[ \hat{x}_6 = x_5 - \left( \frac{1}{a_1} + \frac{1}{d_1} + \frac{1}{a_1} \right) x_6 + \frac{k_1 \tilde{a}_1}{a_1 d_1} M_{in} - \frac{k_2 (\tilde{a}_1 - b_1)}{a_1 d_1} M_{out}. \] (B-6)

Outputs of the systems can be written as follows:

\[ M_{out} = x_2 - \frac{\tilde{a}_1}{b_1 k_2} P_{out} \] (B-15)

\[ M_{in} = x_6 + \frac{c_1}{a_1} P_{in}. \] (B-16)

For known outlet gas pressure and inlet gas flow rate, the state space equations are:

\[ \hat{x}_1 = \frac{-k_1}{k_2 a_1 b_1 c_1 d_1} x_4 + \frac{1}{a_1 b_1 c_1 d_1} \left( 1 - \frac{k_1}{k_2 c_1} \right) M_{in} \] (B-17)

\[ \hat{x}_2 = x_1 - \left( \frac{1}{a_1 b_1 d_1} + \frac{1}{k_2 a_1 b_1 c_1 d_1} \right) x_4 + \left( \frac{1}{a_1 b_1 d_1} + \frac{1}{a_1 c_1 d_1} + \frac{1}{b_1 c_1 d_1} - \frac{2 k_1 \tilde{a}_1^2}{k_2 a_1 b_1 c_1 d_1} \right) M_{in} \] (B-18)

\[ \hat{x}_3 = x_2 - \left( \frac{1}{a_1 b_1 d_1} + \frac{1}{b_1 c_1 d_1} + \frac{k_1 \tilde{a}_1^2}{k_2 a_1 b_1 c_1 d_1} \right) x_4 + \left( \frac{1}{a_1 c_1 d_1} + \frac{1}{b_1 c_1 d_1} - \frac{k_1 \tilde{a}_1^3}{k_2 a_1 b_1 c_1 d_1} \right) M_{in} \] (B-19)

\[ \hat{x}_4 = x_3 - \left( \frac{1}{a_1 b_1 d_1 d_1} + \frac{1}{c_1} M_{in} + \frac{\tilde{a}_1^2}{k_2 b_1 c_1 d_1} P_{out} \right) \] (B-20)

\[ \hat{x}_5 = x_4 - \left( \frac{1}{a_1 b_1 d_1} + \frac{1}{a_1 b_1 d_1} + \frac{1}{a_1 b_1 d_1} + \frac{1}{a_1 b_1 d_1} \right) x_6 \] (B-14)

\[ \hat{x}_6 = x_5 - \left( \frac{1}{a_1} + \frac{1}{d_1} + \frac{1}{a_1} \right) x_6 + c_1 P_{in} - \frac{\tilde{a}_1}{k_2 b_1 c_1 d_1} P_{out}. \]
\[
\begin{align*}
\dot{x}_5 &= -\frac{k_1}{k_2a_1b_2c_1d_1}x_5 + \frac{k_1}{k_2a_1b_1c_1d_1}M_{in} + \frac{k_1a_1}{k_2a_1b_1c_1d_1}P_{out} \quad (B-21) \\
\dot{x}_6 &= x_5 - \left(1 + \frac{2k_1a_1}{k_2a_1b_1c_1d_1}\right)x_8 \\
&\quad + \left(\frac{k_2a_1b_1c_1d_1}{k_2a_1b_1c_1d_1} - 2k_1a_1\right)M_{in} \\
&\quad - \left(\frac{\alpha_1}{k_2a_1b_1c_1d_1} + \frac{k_1a_1^2}{k_2a_1b_1c_1d_1} + \frac{1}{k_2a_1b_1c_1d_1}\right)P_{out} \quad (B-22) \\
\dot{x}_7 &= x_6 - \left(1 + \frac{1}{a_1d_1} + \frac{1}{b_1d_1} + \frac{k_1a_1^2}{k_2a_1b_1c_1d_1}\right)x_8 \\
&\quad + \left(\frac{2k_1a_1}{k_2a_1b_1c_1d_1} + \frac{k_1a_1^2}{k_2a_1b_1c_1d_1}\right)M_{in} \\
&\quad + \left(\frac{\alpha_1}{k_2b_1d_1} + \frac{\alpha_1}{k_2a_1b_1} - \frac{1}{k_2a_1b_1} - \frac{1}{k_2b_1d_1}\right)P_{out} \quad (B-23) \\
\dot{x}_8 &= x_7 - \left(1 + \frac{1}{a_1d_1} + \frac{1}{b_1d_1}\right)x_8 + \frac{k_1a_1^2}{k_2a_1b_1c_1d_1}M_{in} + \left(\frac{\alpha_1}{k_2b_1d_1} - \frac{1}{k_2b_1d_1}\right)P_{out}. \quad (B-24)
\end{align*}
\]

Outputs of the systems can be written as follows

\[
\begin{align*}
P_{in} &= x_4 + \frac{\alpha_1}{c_1}M_{in} \quad (B-25) \\
M_{out} &= x_8 - \frac{\alpha_1}{b_1k_2}P_{out}. \quad (B-26)
\end{align*}
\]

References

